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Published in:

2017 56th Annual Conference of the Society of Instrument and Control Engineers of Japan, SICE 2017

DOI:

[10.23919/SICE.2017.8105543](https://doi.org/10.23919/SICE.2017.8105543)

IMPORTANT NOTE: You are advised to consult the publisher's version (publisher's PDF) if you wish to cite from it. Please check the document version below.

Document Version

Publisher's PDF, also known as Version of record

Publication date:

2017

[Link to publication in University of Groningen/UMCG research database](#)

Citation for published version (APA):

Iori, T., Kawano, Y., & Ohtsuka, T. (2017). Algebraic approach to nonlinear finite-horizon optimal control problems of discrete-time systems with terminal constraints. In *2017 56th Annual Conference of the Society of Instrument and Control Engineers of Japan, SICE 2017* (pp. 220-225). Institute of Electrical and Electronics Engineers Inc.. <https://doi.org/10.23919/SICE.2017.8105543>

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Algebraic Approach to Nonlinear Finite-Horizon Optimal Control Problems of Discrete-Time Systems with Terminal Constraints

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Abstract: This paper proposes a method to solve nonlinear finite-horizon optimal control problems of discrete-time polynomial systems with polynomial terminal constraints. Algebraic equations with all variables at each time step, which are independent of variables at other time steps, are derived from the necessary conditions for optimality by eliminating variables recursively. The candidates of the optimal solution are obtained by solving these equations, and algorithms to find all of these candidates are also proposed. Because of its structure, the proposed method is suitable for nonlinear model predictive control that needs only the initial optimal control law. A simple example to illustrate the methodology and a practical example with the nonlinear model predictive control framework are provided.

Keywords: discrete-time systems, polynomial systems, optimal control, commutative algebra

1. INTRODUCTION

This paper presents a solution method for nonlinear finite-horizon optimal control problems (FHOCs) with terminal constraints for discrete-time systems by restricting a class of nonlinear functions to polynomials. FHOCs with terminal constraints are more general than those without these constraints, and the terminal constraints emerge in many practical situations [1]. In addition, in the nonlinear model predictive control (NMPC) framework, the controller achieves asymptotic stability of the origin by solving FHOCs with the zero-state terminal constraint [2, 3]. However, unlike FHOCs without terminal constraints, Lagrange multipliers associated with the terminal constraints are added to the optimization problem as variables, and they induce additional difficulties when solving problems. Some of them, such as linear quadratic control problems with a fixed terminal state, can be solved analytically [4]. However, solving them this way in general cases is extremely difficult, and various numerical methods have been proposed in the literature [1, 5, 6].

Recently, the recursive elimination method has been proposed for solving FHOCs of discrete-time rational systems [7]. This method analytically constructs algebraic equations of the state, input, and costate at each time step, each of which is independent of the variables at other time steps. Moreover, these algebraic equations can be considered as an implicit function representation of the optimal solution under some conditions. Since these algebraic equations are computed symbolically, once they are obtained, they can be solved for any initial state by a numerical method such as Newton's method. This feature is the advantage of the recur-

sive elimination method, particularly when it is utilized in the NMPC framework. In NMPC, an FHOC needs to be solved in real time, and only the initial optimal input is used as an actual input to the system. By utilizing the recursive elimination method, the initial optimal input can be obtained by solving the algebraic equations at the only initial time step that can be obtained in offline computation in advance. The computational cost of solving them is much lower than that of solving the whole of an FHOC, but, FHOCs with terminal constraints were not considered [7].

For FHOCs of polynomial systems with polynomial constraints, various solution methods have been proposed. Constrained FHOCs have been formulated as parametric optimization problems depending on the initial state and solved by applying cylindrical algebraic decomposition (CAD) [8]. Moreover, in this article, the candidates of the optimal solution were computed from the Karush-Kuhn-Tucker (KKT) conditions by applying the notion of zero-dimensional ideals in commutative algebra. The sum-of-square decomposition is also useful to solve parametric optimizations [9], and its applications to NMPC have been discussed elsewhere [10]. However, the structure of the optimal control problem was not considered in these methods. Iwane et al. [11] solved constrained FHOCs by dynamic programming combined with CAD, in which the value functions and optimal feedback control laws can be found by solving the optimization problem at each step by CAD. However, the CAD algorithm is too general to solve constrained FHOCs, and it requires extremely heavy computational burdens.

This paper proposes the recursive elimination method for FHOCs of discrete-time polynomial systems with polynomial terminal constraints. By utilizing the elimination ideal in commutative algebra, the proposed

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method derives algebraic equations in the state, input, and costate at each time step and the Lagrange multiplier associated with the terminal constraint, which are independent of variables at other time steps. Solutions of these algebraic equations include the optimal solution. By solving them for a given initial state recursively, candidates of the optimal states, inputs, costates, and Lagrange multiplier can be obtained. Moreover, by eliminating costates and the Lagrange multiplier from the algebraic equations at the initial time step, algebraic equations in the initial state and input are obtained, which can be considered as an implicit function representation of the optimal feedback controller.

Notation: Subscript k denotes the time step of discrete-time systems throughout this paper, while i denotes the components of vectors. For example, x_k denotes a vector at time step k , while x_{ki} denotes the i th component of the vector x_k . To avoid confusion, x_{ki} is also denoted by $x_{k,i}$ if necessary. For vectors $X = [X_1 \cdots X_n]^T$ and $Y = [Y_1 \cdots Y_m]^T$, $\mathbf{R}[X, Y]$ denotes the ring of polynomials in the components of X and Y over the field of real numbers $\mathbf{R}$. An ideal generated by the set of polynomials $\{f_1, \dots, f_s\} \subset \mathbf{R}[X]$ is defined as $\langle f_1, \dots, f_s \rangle := \{a_1 f_1 + \cdots + a_s f_s \mid a_1, \dots, a_s \in \mathbf{R}[X]\}$, and the set $\{f_1, \dots, f_s\}$ is called generators of the ideal. For an ideal $I \subset \mathbf{R}[X]$, $\mathcal{V}(I)$ denotes the algebraic set defined by I , which is the set of common zeros of all polynomials in I . In this paper, we are only concerned with zeros in $\mathbf{R}^n$. That is, $\mathcal{V}(I) \subset \mathbf{R}^n$. If an ideal I is generated by $\{f_1, \dots, f_s\}$, $\mathcal{V}(I)$ equals the set of common zeros of $\{f_1, \dots, f_s\}$ [12]. In this case, $\mathcal{V}(I)$ is also denoted by $\mathcal{V}(f_1, \dots, f_s)$. For a scalar-valued function $V(X)$, $\nabla_X V$ denotes the column vector consisting of the partial derivatives of V with respect to X_i ($i = 1, 2, \dots, n$). That is, $\nabla_X V = [\partial V / \partial X_1 \cdots \partial V / \partial X_n]^T$. For a scalar-valued function $V(X_k)$ of variables depending on k , we use $\nabla_X V(X_k)$ instead of $\nabla_{X_k} V(X_k)$.

2. PROBLEM FORMULATION

Consider the following FHOC with terminal constraints for a given terminal time step $N \in \mathbf{Z}_+ = \{0, 1, 2, \dots\}$:

$$\min_{u_1, \dots, u_{N-1}} \phi(x_N) + \sum_{k=0}^{N-1} L_k(x_k, u_k), \quad (1)$$

$$\text{subject to } x_{k+1} = f_k(x_k, u_k) \quad \text{for } k = 0, \dots, N-1, \quad (2)$$

$$x_0 = \bar{x}, \quad (3)$$

$$\psi(x_N) = 0, \quad (4)$$

where $x_k \in \mathbf{R}^n$ and $u_k \in \mathbf{R}^m$ denote the state and input of a system, respectively, with $k \in \mathbf{Z}_+$, and $\bar{x}$ is a given value of the initial state. The scalar-valued functions $\phi: \mathbf{R}^n \rightarrow \mathbf{R}$ and $L_k: \mathbf{R}^n \times \mathbf{R}^m \rightarrow \mathbf{R}$ denote a terminal cost and stage costs, respectively. A set of constraints (2) is the state equation of the system, and each $f_k: \mathbf{R}^n \times \mathbf{R}^m \rightarrow \mathbf{R}^n$ is a vector-valued function. A set of terminal

constraints is defined as (4) where $\psi: \mathbf{R}^n \rightarrow \mathbf{R}^l$ is a vector-valued function of x_N for $l \in \mathbf{Z}_+$. Throughout this paper, all of the functions, ϕ, L_k, f_k, ψ , in (1)–(4) are supposed to be polynomial functions.

For the constrained optimization problem (1)–(4), the stationary conditions are considered. For convenience when describing the conditions, let H_k be the discrete-time Hamiltonian at time step k defined by

$$H_k(x_k, u_k, p_{k+1}) := L_k(x_k, u_k) + p_{k+1}^T f_k(x_k, u_k), \quad (5)$$

where $p_k \in \mathbf{R}^n$ denotes the costate (or adjoint variable). The stationary conditions are obtained as the discrete-time Euler-Lagrange equations (ELEs) for $k = 0, \dots, N-1$ [1]:

$$f_k(x_k, u_k) - x_{k+1} = 0, \quad (6)$$

$$\nabla_x H(x_k, u_k, p_{k+1}) - p_k = 0, \quad (7)$$

$$\nabla_u H(x_k, u_k, p_{k+1}) = 0, \quad (8)$$

$$\left(\frac{\partial \psi(x_N)}{\partial x} \right)^T \nu + \nabla_x \phi(x_N) - p_N = 0, \quad (9)$$

$$\psi(x_N) = 0, \quad (10)$$

where $\nu \in \mathbf{R}^l$ denotes the Lagrange multiplier associated with the terminal constraint (4). By solving ELEs (6)–(10), the candidates of the optimal solution for problem (1)–(4) are obtained.

Since the initial state is given as $x_0 = \bar{x}$ and the terminal state x_N is constrained by (10), the ELEs can be considered as a two-point boundary-value problem (TP-BVP). When a state equation (6) and an adjoint equation (7) involve nonlinear functions, it is very difficult to solve the ELEs analytically. It is also difficult to solve the ELEs even numerically because its iterative solution methods require heavy computational burdens. In the proposed method, the ELEs are analytically decomposed to a sequence of algebraic equations. Each of them involves the state x_k , control input u_k , and costate p_k at each time step k and the Lagrange multiplier ν , which is independent of variables at other time steps. Because of this structure, these equations can be solved independently at each time step; thus, it takes lower computational costs than solving the whole of the ELEs. Since ϕ, L_k, f_k , and ψ are polynomials, some mathematical tools of commutative algebra and algebraic geometry can be used to decompose them. Decomposed algebraic equations are satisfied at solutions of the ELEs. In other words, these solutions are regarded as candidates of the optimal solution. Moreover, by eliminating p_k and ν from these polynomials, algebraic equations in x_k and u_k can be obtained, which can be considered as an implicit function representation of the optimal feedback controller.

3. RECURSIVE ELIMINATION METHOD

ELEs (6)–(10) can be viewed as algebraic equations in sequences of the states $(x_k)_{k=0}^N$, costates $(p_k)_{k=0}^N$, and inputs $(u_k)_{k=0}^{N-1}$ and the Lagrange multiplier ν . By eliminating the variables, except those at time step k , from the ELEs, algebraic equations in x_k , p_k , u_k , and ν can be obtained. For FHOCs without terminal constraints, a method to obtain the algebraic equations that are satisfied at optimal states, inputs, and costates is proposed [7]. This method is called the ‘recursive elimination method’ and uses an elimination ideal in commutative algebra to eliminate variables from the ELEs recursively. In this paper, this method is applied to FHOCs with terminal constraints. Note that the Lagrange multiplier associated with terminal constraints, which does not appear in FHOCs without constraints, has to be considered.

Before introducing the algorithm of the method for FHOCs with terminal constraints, some tools from commutative algebra and algebraic geometry used in the algorithm are introduced. For an ideal $I \subset \mathbf{R}[X, Y]$ with $X = [X_1 \cdots X_n]^T$ and $Y = [Y_1 \cdots Y_m]^T$, we can consider a set of the polynomials in I that involve only the variables Y . This set $I \cap \mathbf{R}[Y]$ also becomes an ideal [12] and is called the *elimination ideal* of I with respect to X [13]. The relation between the algebraic set defined by $I \cap \mathbf{R}[Y]$ and $\mathcal{V}(I)$ is characterized by the following lemma [12]:

Lemma 1: For an ideal $I \subset \mathbf{R}[X, Y]$, $\pi_Y(\mathcal{V}(I)) \subset \mathcal{V}(I \cap \mathbf{R}[Y])$ holds, where $\pi_Y: \mathbf{R}^n \times \mathbf{R}^m \rightarrow \mathbf{R}^m$ defined by $(X, Y) \mapsto Y$ is the projection of $\mathcal{V}(I)$ onto the Y -space, i.e., $\pi_Y(\mathcal{V}(I)) = \{Y \mid \exists X \text{ s.t. } (X, Y) \in \mathcal{V}(I)\}$.

Since generators of $I \cap \mathbf{R}[Y]$ are polynomials of Y while those of I are polynomials of X and Y , the calculation of the elimination ideal $I \cap \mathbf{R}[Y]$ corresponds to the elimination of variables X from the simultaneous equations of X and Y . Generators of an elimination ideal $I \cap \mathbf{R}[Y]$ can be computed from generators of I by using a Gröbner basis [12]. A Gröbner basis is a set of generators that has some good properties, and its computation algorithm is implemented in various symbolic computation systems such as Mathematica or Maple.

From Lemma 1, Algorithm 1 to get a set of algebraic equations in x_k , u_k , p_k , and ν at each time step k is obtained.

Theorem 1: Suppose that the sequences of the states $(\hat{x}_k)_{k=0}^N$, inputs $(\hat{u}_k)_{k=0}^{N-1}$, and costates $(\hat{p}_k)_{k=0}^N$ and the Lagrange multiplier $\hat{\nu}$ are the optimal solution to problem (1)–(4). Then, for the ideals J_k ($k = 0, \dots, N$) in Algorithm 1, the following statements hold:

$$(\hat{x}_N, \hat{p}_N, \hat{\nu}) \in \mathcal{V}(J_N), \quad (11)$$

$$(\hat{x}_k, \hat{p}_k, \hat{u}_k, \hat{\nu}) \in \mathcal{V}(J_k) \quad (k = 0, \dots, N-1). \quad (12)$$

That is,

$$F_N(\hat{x}_N, \hat{p}_N, \hat{\nu}) = 0, \quad (13)$$

$$F_k(\hat{x}_k, \hat{p}_k, \hat{u}_k, \hat{\nu}) = 0 \quad (k = 0, \dots, N-1). \quad (14)$$

Algorithm 1 Recursive Elimination Method for FHOC with Terminal Constraints

Input: ELEs (6)–(10) for FHOC with terminal constraints

Output: Algebraic equations $F_k = 0$ ($k = 0, \dots, N$)

- 1: Let F_N be set of polynomials consisting of left-hand sides of (9) and (10), and let $J_N := \langle F_N \rangle \subset \mathbf{R}[x_N, p_N, \nu]$, $k := N - 1$
- 2: **while** $k \geq 0$ **do**
- 3: $\bar{I}_k := \langle f_k(x_k, u_k) - x_{k+1}, \nabla_x H_k(x_k, u_k, p_{k+1}) - p_k, \nabla_u H_k(x_k, u_k, p_{k+1}), F_{k+1} \rangle$
- 4: $J_k := \langle F_k \rangle = \bar{I}_k \cap \mathbf{R}[x_k, p_k, u_k, \nu]$
- 5: $k \leftarrow k - 1$
- 6: **end while**

Proof : The proof is by induction. First, at $k = N$, the ideal J_N is generated by the left-hand sides of ELEs (9) and (10), and then the statement (11) holds. Suppose that $(\hat{x}_k, \hat{p}_k, \hat{u}_k, \hat{\nu}) \in \mathcal{V}(J_k)$ at time step k , and from the definition of $\bar{I}_{k-1}$, we have

$$(\hat{x}_{k-1}, \hat{p}_{k-1}, \hat{u}_{k-1}, \hat{x}_k, \hat{p}_k, \hat{u}_k, \hat{\nu}) \in \mathcal{V}(\bar{I}_{k-1}). \quad (15)$$

By projecting $\mathcal{V}(\bar{I}_{k-1})$ onto the $(x_{k-1}, p_{k-1}, u_{k-1}, \nu)$ -space and by applying Lemma 1,

$$\begin{aligned} &(\hat{x}_{k-1}, \hat{p}_{k-1}, \hat{u}_{k-1}, \hat{\nu}) \\ &\in \pi_{(x_{k-1}, p_{k-1}, u_{k-1}, \nu)}(\mathcal{V}(\bar{I}_{k-1})) \subset \mathcal{V}(J_{k-1}) \end{aligned} \quad (16)$$

is obtained, and the proof is completed by induction. ■

Theorem 1 shows that all polynomials in F_k must vanish at the optimal solution of $(\hat{x}_k)_{k=0}^N$, $(\hat{p}_k)_{k=0}^N$, $(\hat{u}_k)_{k=0}^{N-1}$ and $\hat{\nu}$, which can be regarded as the necessary conditions for optimality. Moreover, the following corollary is readily shown from Lemma 1 and Theorem 1.

Corollary 1: Suppose that the sequences of $(\hat{x}_k)_{k=0}^N$ and $(\hat{u}_k)_{k=0}^{N-1}$ and $\hat{\nu}$ are the optimal solution of problem (1)–(4), and let ideals K_k, W_k be defined by

$$K_k := J_k \cap \mathbf{R}[x_k, u_k, \nu], \quad W_k := J_k \cap \mathbf{R}[x_k, \nu].$$

Then, the following statements hold.

$$(\hat{x}_k, \hat{u}_k, \hat{\nu}) \in \mathcal{V}(K_k), \quad (\hat{x}_k, \hat{\nu}) \in \mathcal{V}(W_k).$$

If an initial state is given as $x_0 = \bar{x}$, by substituting $\bar{x}$ into the generators of W_0 , a set of polynomials of ν is obtained. From Corollary 1, the common zeros of these polynomials include the optimal Lagrange multiplier $\hat{\nu}$. In other words, common zeros of these polynomials can be regarded as candidates of $\hat{\nu}$. For the following discussion, candidates of the optimal value $\hat{y}$ of variable y are denoted by $\tilde{y}$. For example, the common zeros of polynomials obtained by substituting $\bar{x}$ into the generators of W_0 are denoted by $\tilde{\nu}$. Also, $\tilde{u}_0$ can be obtained by substituting $\bar{x}$ and $\tilde{\nu}$ into the generators of K_0 and computing their common zeros. Then, $\tilde{x}_1$ is obtained by substituting $\bar{x}$ and $\tilde{u}_0$ into state equation (2). In the same way, $\tilde{u}_1$ can be derived from $\tilde{x}_1, \tilde{\nu}$, and K_1 , and $\tilde{x}_2$ can be obtained by

substituting $\tilde{x}_1$ and $\tilde{u}_1$ into equation (2). In summary, $\tilde{v}$, $(\tilde{x}_k)_{k=0}^N$, and $(\tilde{u}_k)_{k=0}^{N-1}$ can be derived from x_0 , W_0 , and $(K_k)_{k=0}^{N-1}$ as follows:

1. **Step 1:** Compute $\tilde{v}$ from polynomials obtained by substituting $x_0 = \bar{x}$ into the generators of W_0 .
2. **Step 2:** Compute $\tilde{u}_k$ from the generators of K_k by substituting $\tilde{x}_k$ and $\tilde{v}$.
3. **Step 3:** Compute $\tilde{x}_{k+1}$ from equation (2) by substituting $\tilde{x}_k$ and $\tilde{u}_k$.
4. **Step 4:** Repeat Step 2 and Step 3 from $k = 0$ to $N - 1$ recursively.

In addition, the sequence of $(\tilde{p}_k)_{k=0}^N$ can be obtained by substituting the sequences of $(\tilde{x}_k)_{k=0}^N$ and $(\tilde{u}_k)_{k=0}^{N-1}$ into (7) and (9).

Note that the value of $\tilde{v}$ computed from W_0 and $\tilde{u}_k$ computed from K_k may not be determined uniquely because the generators of these ideals are often polynomials of degree 2 or higher, which may have several different roots. That is, several values of $\tilde{v}$ or $\tilde{u}_k$ may be obtained from W_0 or K_k . To compute all values of candidates of $\tilde{v}$, $(\tilde{u}_k)_{k=0}^{N-1}$, and $(\tilde{x}_k)_{k=0}^N$, the candidates of state and input at the next step should be computed for each value of $\tilde{v}$ or $\tilde{u}_k$. Therefore, the tree identified by each value of $\tilde{v}$ is suitable for an expression of all values. In this tree, the root node is associated with an initial state x_0 , and the other nodes are associated with the values of $\tilde{u}_k$ and $\tilde{x}_{k+1}$. Let $T_{\tilde{v}}$ be a tree corresponding to a value of $\tilde{v}$, and let $\mathcal{T}$ be a set of $T_{\tilde{v}}$ for all values of $\tilde{v}$. The computation of $\mathcal{T}$ consists of two algorithms: Algorithms 2 and 3. In these algorithms, the values of candidates of state and input associated with node v are denoted by $x(v)$ and $u(v)$, respectively.

First, the recursive computation of candidates of input $\tilde{u}_k$ and state $\tilde{x}_{k+1}$ for a value of $\tilde{v}$ is realized in Algorithm 2. By using this algorithm, a tree $T_{\tilde{v}}$ can be obtained for a given initial state $\bar{x}$ and a value of $\tilde{v}$. A set of trees $\mathcal{T}$ can be obtained by Algorithm 3. If the number of values of $(\tilde{u}_k)_{k=0}^{N-1}$ and $\tilde{v}$ is finite, all values of $(\tilde{x}_k)_{k=0}^N$ and $(\tilde{u}_k)_{k=0}^{N-1}$ can be obtained by Algorithm 3. The value of $\tilde{p}_N$ can be computed by substituting $\tilde{x}_N$ and $\tilde{v}$ into (9), and the values of $\tilde{p}_k$ can be obtained by substituting the sequences of $(\tilde{x}_k)_{k=0}^N$ and $(\tilde{u}_k)_{k=0}^{N-1}$ and $\tilde{p}_N$ into (7), recursively. From Lemma 1, some values of $\tilde{x}_k$, $\tilde{u}_k$, and $\tilde{p}_{k+1}$ may not satisfy equation (8), and some value of $\tilde{x}_N$ may not satisfy terminal constraint (10). By excluding nodes associated with these invalid values from $T_{\tilde{v}}$, the number of candidate values can be reduced. Finally, if the feasible set defined by (2)–(4) is compact, the global optimal solution can be obtained by comparing the values of cost function (1) for all sequences of values of $\tilde{x}_k$ and $\tilde{u}_k$.

Algorithm 2 Recursive Computation of Candidates $\tilde{u}_k$, $\tilde{x}_{k+1}$

Input: Sequence of ideals $(K_j)_{j=0}^{N-1}$, node v , value of candidate of optimal Lagrange multiplier $\tilde{v}$, time step k

Output: Tree that has node v as its root node

```

1: function CALCCANDS( $(K_j)_{j=0}^{N-1}$ ,  $v$ ,  $\tilde{v}$ ,  $k$ )
2:   Let  $\tilde{U}$  be set of common zeros that is obtained by
   substituting  $x(v)$  ( $= \tilde{x}_k$ ) and  $\tilde{v}$  into generators of
    $K_k \in (K_j)_{j=0}^{N-1}$ 
3:   for each  $\tilde{u}_k \in \tilde{U}$  do
4:      $\tilde{x}_{k+1} := f_k(x(v), \tilde{u}_k)$ 
5:     Add new node  $v'$  as child node of  $v$ 
6:      $x(v') \leftarrow \tilde{x}_{k+1}$ 
7:      $u(v') \leftarrow \tilde{u}_k$ 
8:     if  $k \neq N - 1$  then
9:       CALCCANDS( $(K_j)_{j=0}^{N-1}$ ,  $v'$ ,  $\tilde{v}$ ,  $k + 1$ )
10:    end if
11:  end for
12:  return Tree that has root node  $v$ 
13: end function

```

Algorithm 3 Computation of All Candidates of Solution from Sequences of Ideals

Input: Ideal W_0 , sequence of ideals $(K_j)_{j=0}^{N-1}$, initial state $\bar{x}$

Output: Set of trees $\mathcal{T}$ that consists of trees $T_{\tilde{v}}$ for all values of candidate $\tilde{v}$

```

1: Let  $\tilde{N}$  be set of common zeros of polynomials that
   are obtained by substituting  $\bar{x}$  into generators of  $W_0$ 
2: for each  $\tilde{v} \in \tilde{N}$  do
3:    $x(v_{\tilde{v}}^{\text{root}}) \leftarrow \bar{x}$ 
4:    $u(v_{\tilde{v}}^{\text{root}}) \leftarrow 0$ 
5:    $T_{\tilde{v}} := \text{CALCCANDS}((K_j)_{j=0}^{N-1}, v_{\tilde{v}}^{\text{root}}, \tilde{v}, 0)$ 
6:   Add  $T_{\tilde{v}}$  to  $\mathcal{T}$ 
7: end for

```

4. NUMERICAL EXAMPLES

4.1. Academic Example

The proposed algorithm is illustrated in an example. Consider the following FHOC with a terminal constraint:

$$\min_{u_1, \dots, u_{N-1}} \quad \frac{1}{2} x_{N,1}^2 + \sum_{k=0}^{N-1} \frac{1}{2} u_k^2, \quad (17)$$

subject to

$$\begin{cases} x_{k+1,1} = & x_{k,1} + x_{k,2} \\ x_{k+1,2} = & x_{k,2} + x_{k,1}^2 + u_k, \end{cases} \quad (18)$$

$$x_0 = \bar{x}, \quad (19)$$

$$x_{N,2} = 0. \quad (20)$$

For $N = 3$, Algorithm 1 yields a sequence of ideals

J_0, J_1, J_2 , and J_3 as follows:

$$J_0 = \langle 2u_0x_{01} + p_{01} - p_{02} - u_0, \dots \rangle, \quad (21)$$

$$J_1 = \langle 2u_1x_{11} + p_{11} - p_{12} - u_1, x_{11}^2 + \nu + 2u_1 + x_{11} + 2x_{12}, \dots \rangle, \quad (22)$$

$$J_2 = \langle u_2 + \nu, x_{21} - u_2 + x_{22} - p_{22}, x_{21}^2 + p_{22} + 2u_2 - x_{21}, \dots \rangle, \quad (23)$$

$$J_3 = \langle x_{32}, p_{32} - \nu, x_{31} - p_{31} \rangle. \quad (24)$$

Some terms and polynomials in the generators are omitted due to space limitations. From ideals J_0, J_1 , and J_2 , ideal W_0 and a sequence of ideals $(K_k)_{k=0}^2$ are obtained as

$$W_0 = \langle -12\nu^3 + 4(x_{01}^2 + 2x_{01}x_{02} + x_{02}^2 - x_{01} - x_{02} - 9)\nu^2 + \dots \rangle, \quad (25)$$

$$K_0 = \langle 2u_0^2 + 4(x_{01}^2 + x_{01} + 2x_{02})u_0 + 2x_{01}^4 + 4x_{01}^3 + 8x_{01}^2x_{02} + \dots, \dots \rangle, \quad (26)$$

$$K_1 = \langle 2u_1 + \nu + x_{11}^2 + x_{11} + 2x_{12}, -3u_1 - 3\nu + 2x_{11}x_{12} + x_{12}^2 - 2x_{11} - x_{12} \rangle, \quad (27)$$

$$K_2 = \langle u_2 + \nu, u_2 + x_{21}^2 + x_{22} \rangle. \quad (28)$$

In this case, for initial states of $x_0 = [1 \ 3]^T$, $[3 \ -2]^T$, $[-1 \ 2]^T$, and $[-2 \ -2]^T$, the values of ν, u_0, u_1 , and u_2 are determined uniquely. Figure 1 shows the trajectories of the system (18) with feedback control inputs derived from K_0, K_1 , and K_2 . It is readily observed that all of the terminal states reach the line of $x_2 = 0$. That is, terminal constraint (20) is satisfied.

Table 1 compares the computational times required to obtain inputs from the generators of ideals K_k and W_k with those from the whole of the ELEs for an initial state of $[3 \ -2]^T$. Newton's method is implemented for numerical computation using Maple on a PC (CPU: Intel Core i7-4650U 1.70 GHz, RAM: 8.00 GB), and the inverted Jacobian is obtained explicitly in this example. Since the computational times depend on the initial guess of Newton's method, they are averaged for 1000 randomly chosen initial points for each case in Table 1. For the time

Fig. 1 Trajectories for some initial states.

Table 1 Computational times [ms] of proposed method and conventional method for time steps.

	$k = 0$	$k = 1$	$k = 2$
Generators of K_k	0.24	0.05	0.05
Whole of ELEs	12.97	0.30	0.03

steps of $k = 0$ and 1, the computational times of the proposed method are remarkably smaller than those for solving the whole of the ELEs, which shows the efficiency of the proposed algorithm.

4.2. NMPC Application

In more practical situations, an FHOCP is often combined with the NMPC framework, in which only the initial optimal input $\hat{u}_0$ is required. By eliminating ν from an ideal $K_0 \subset \mathbf{R}[x_0, u_0, \nu]$, a set of polynomials $K'_0 \subset \mathbf{R}[x_0, u_0]$ is obtained as generators of an elimination ideal, which can be considered as an implicit function representation of the optimal feedback controller at time step $k = 0$. Therefore, in the NMPC framework, a control input can be computed from K'_0 by some numerical computation algorithm such as Newton's method. Note that, if the FHOCP solved at each step in NMPC has the terminal constraint of $x_N = 0$, the origin is asymptotically stable with a region of attraction that is a set of initial states for which the FHOCP has a solution [2].

For a practical example, the stabilization of an inverted pendulum by NMPC is considered. A continuous-time equation of motion is defined as follows:

$$\ddot{\theta}(t) + \dot{\theta}(t) - \sin \theta(t) = u(t), \quad (29)$$

where $t \in \mathbf{R}_+ = [0, \infty)$ denotes the continuous time, $\theta(t) \in \mathbf{R}$ denotes the angle subtended by the rod and the vertical axis, and $u(t) \in \mathbf{R}$ denotes the controllable external torque. By using forward difference approximation with a sampling period of $\Delta t = 0.05$ and approximating $\sin(\theta)$ as $s(\theta) := \theta - \frac{1}{6}\theta^3$ to apply the proposed method, the discrete-time model can be obtained as

$$\begin{cases} x_{k+1,1} = x_{k,1} + x_{k,2}\Delta t \\ x_{k+1,2} = x_{k,2} + (-x_{k,2} + s(x_{k,1}) + u_k)\Delta t, \end{cases} \quad (30)$$

where $x_{k,1}$ and $x_{k,2}$ denote the angle and angular velocity, respectively, and u_k denotes the controllable external torque. The control objective is to regulate the state to the origin and reduce control burdens. Therefore, the stage costs and terminal constraint are defined as

$$L_k(x_k, u_k) = \frac{1}{2}u_k^2, \quad \psi(x_N) = x_N = 0. \quad (31)$$

Due to this terminal constraint, the terminal cost is omitted ($\phi(x_N) \equiv 0$).

For $N = 2$, K'_0 consists of only one polynomial:

$$\kappa_0(x_0, u_0) = x_{01}^3 - 6u_0 - 2406x_{01} - 234x_{02}. \quad (32)$$

Since $\kappa_0(x_0, u_0)$ is linear in u_0 , the equation

$\kappa_0(x_0, u_0) = 0$ can be solved analytically for u_0 , and the optimal feedback controller can be obtained explicitly.

Figure 2 shows the trajectories of the system (29) with the feedback controller obtained from $\kappa_0(x_0, u_0)$ and free response without the controller. The initial state is set to $x_0 = [0.5 \ 0]^T$. In the controlled case, the state and control input are sampled and computed with a sampling period of 0.01 time units. That is, the sampling period of NMPC is smaller than that of the discretization. It is readily observed that the state of the controlled system is regulated to the origin.

Fig. 2 Trajectories of system (29) derived from controlled system (solid line) and its free response (dashed line).

5. CONCLUSION AND FUTURE WORK

In this paper, the recursive elimination method for FHOCs of discrete-time polynomial systems with polynomial terminal constraints is proposed. By applying the elimination ideal in commutative algebra, algebraic equations at each time step are obtained from the ELEs. For a given initial state, these algebraic equations can be solved recursively from the initial step without iterative computation, and all of the candidates of the optimal solution can be obtained as the trees identified by values of the candidate of a Lagrange multiplier. The class of problems that can be treated by the proposed method is easily expanded to the class with rational systems and rational terminal constraints. For further research, we plan to explore the condition for finiteness of candidates of the optimal solution. Otherwise, it is necessary to find the optimal solution even if there exists an infinite number of candidates.

ACKNOWLEDGMENTS

This work was partly supported by JSPS KAKENHI Grant Numbers 15K18087 and 15H02257 and JST CREST Grant Number JPMJCR15K2.

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