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Data-Driven Regret Minimization in Routing Games under Uncertainty

Jonathan Dietrich, Ashish R. Hota, and Ashish Cherukuri

Abstract—This paper studies network routing under uncertain costs. We introduce the notion of regret and present methods to minimize it using data. Given a flow vector and a realization of the uncertainty, the regret experienced by a user on a particular path is the difference between the cost incurred on the path and the minimum cost across all paths connecting the same origin and destination. The network-wide regret is the cumulative regret experienced by all agents. We show that, for a fixed uncertainty, the total regret of all agents is a convex function provided the cost function of each path is affine and monotone. We provide two data-driven methods that minimize the expected value and a specified quantile of the total regret, respectively. Simulations compare our solutions to existing approaches of handling uncertainty in routing games.

I. INTRODUCTION

In modern transportation systems, a large number of users simultaneously interact over a shared network, and choose their routes in a selfish and decentralized manner. Therefore, their behavior is often studied in the framework of game theory. In a nonatomic routing game, each user or agent is treated as an infinitesimal entity who chooses a path to travel from a desired source to a destination in order to minimize her travel time or cost; the travel cost depends on the route choices of all users or the aggregate traffic flow. Furthermore, route choice of any individual user has little impact on the aggregate traffic flow. In this setting, a *Wardrop equilibrium* (WE) corresponds to a traffic flow where no agent can reduce her cost by unilaterally changing her route [1].

Due to its underlying behavioral foundation, WE is widely studied in the literature vis-a-vis designing algorithms for computing a WE [2], [3], [4], characterizing the inefficiency of WE [5], [6], designing tolls to reduce congestion [7], and planning for future transportation systems [8]. However, in practice, transportation systems operate under a significant amount of uncertainty resulting from accidents, weather fluctuations, and construction work, among others. The uncertainty results in variability in the travel time and demand. Consequently, there is no single traffic flow that constitutes a WE for every realization of the uncertain parameters.

Therefore, it is challenging to define a notion of ‘equilibrium’ flow in presence of uncertainty. Existing literature has mostly relied on solving a stochastic nonlinear complementarity problem (NCP) [9], either in an expectation based [10], [11] or in a robust manner [12], [13] (see Section III for more details). These approaches require the decision maker to know either the distribution or the support of the

uncertain parameters. However, in practice, this information is not known with certainty, but instead, the decision maker has access to samples or realizations of the uncertainty. Furthermore, the flows computed by the above methods lack suitable behavioral justification.

In practice, it is unlikely that users change their routes frequently based on different realizations of uncertainty. In fact, a recent large-scale field study on route choice by users in Singapore [14] showed that the percentage of users selecting the same route (for a given origin and destination nodes) consistently is very high (in the order of 94%). Furthermore, the authors define a measure, referred to as the *imitation regret* of a user as the difference in travel time of the user and the lowest travel time among a set of users commuting between the same origin and destination nodes. They observed that the paths chosen consistently by the users have relatively small imitation regret for most realizations of uncertainty. Motivated by these empirical findings, the goal of this paper is to formally define the notion of regret in routing games under uncertainty, and propose tractable data-driven techniques for computing flows with suitable performance vis-a-vis aggregate regret.

We consider routing games where the demand is fixed and the travel costs depend on the traffic flow and uncertain parameters. For a given realization of uncertainty, we define the regret experienced by a user as the difference between the travel cost along the path chosen by the user and the minimum travel cost among all paths between the same origin destination pair. The aggregate regret of all users is defined as the sum of regret on each individual path weighted by the flow on that path. We prove that the aggregate regret is convex in traffic flow for every realization of uncertainty when costs are affine and monotone in the traffic flow.

We then present two data-driven methods for computing traffic flows under uncertainty. We first consider a distributionally robust approach where we minimize the expected regret with respect to the worst case distribution among a set of distributions termed ambiguity set; in particular, all distributions within a certain distance (defined via the Wasserstein metric) to the empirical distribution induced by the data or samples of uncertainty. We choose this class of data-driven ambiguity sets as they have several advantages, e.g., if the data is generated from an underlying distribution, then as the number of samples increases, the true distribution lies in this ambiguity set with high probability; see [15], [16] for further details. When the cost functions are affine in traffic flow and uncertainty, we show that we can minimize the expected regret over Wasserstein ambiguity sets by leveraging a tractable reformulation developed in [15].

We then formulate a chance constrained optimization

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problem to compute a flow and the smallest bound such that the aggregate regret for this flow is smaller than this bound with high probability. A similar notion of equilibrium was hypothesized in [14] without any quantitative analysis. Our result on the convexity of regret enables us to solve this program in a tractable and data-driven manner via the scenario approach [17], [18]. We compare the proposed data-driven approaches with the existing expectation based and robust approaches via extensive simulations.

II. ROUTING GAME AND WARDROP EQUILIBRIUM

Let $\mathcal{G} = (\mathcal{N}, \mathcal{A})$ be a directed graph that represents the *traffic network*, with \mathcal{N} being the set of *nodes* and $\mathcal{A} \subseteq \mathcal{N} \times \mathcal{N}$ being the set of *arcs*, also referred as *links*. The graph can, for instance, model intersections as nodes and streets as arcs in a city. The sets of *origin* and *destination* nodes are denoted by \mathcal{O} and \mathcal{D} , respectively. The set of *origin-destination (OD) pairs* is $\mathcal{W} \subseteq \mathcal{O} \times \mathcal{D}$. Let \mathcal{P}_w denote the set of available *paths* for the OD pair $w \in \mathcal{W}$ and let $\mathcal{P} = \cup_{w \in \mathcal{W}} \mathcal{P}_w$ be the set of all OD paths, with $|\mathcal{P}| = K$. We assume that there exists at least one path between any OD pair.

We consider the nonatomic routing game where each individual agent or player is infinitesimal and has negligible impact on the aggregate traffic flow. Each agent is associated with an OD pair $w \in \mathcal{W}$, and chooses a path $p \in \mathcal{P}_w$ in a selfish and noncooperative manner. The route choices of all agents give rise to the aggregate traffic which is modeled as a continuous nonnegative *flow vector* $h \in \mathbb{R}_+^K$ with h_p being the *flow* on a path $p \in \mathcal{P}$. Agents who choose path $p \in \mathcal{P}$ experience a nonnegative *cost* (e.g., travel time or delay) $C_p : \mathbb{R}_+^K \rightarrow \mathbb{R}_+$, $h \mapsto C_p(h)$, that is a function of the aggregate traffic. In other words, the cost on a given path potentially depends on the flows on the entire network. We denote the minimum cost for OD pair $w \in \mathcal{W}$ by the function $h \mapsto C_w^{\min}(h) := \min_{p \in \mathcal{P}_w} C_p(h)$.

In addition to being nonnegative, the aggregate flow between each OD pair must satisfy the travel demand. We denote the *demand* for OD pair $w \in \mathcal{W}$ by $d_w \in \mathbb{R}_+$. Collecting the above described notions, a *network routing game* is defined by the tuple $(\mathcal{G}, \mathcal{W}, \mathcal{P}, C, d)$. The definition is quite general and can incorporate multiple modes of transit, multiple classes of users, interactions between traffic on different arcs and limited choices of paths between OD pairs [19]. The notion of equilibrium for network routing games is that of a Wardrop equilibrium (WE) [1]. A flow vector $h^* \in \mathbb{R}_+^K$ is called a *Wardrop equilibrium (WE)* for a network routing game $(\mathcal{G}, \mathcal{W}, \mathcal{P}, C, d)$ if: (i) h^* satisfies the demand for all OD pairs and (ii) for any OD pair w , a path $p \in \mathcal{P}_w$ has nonzero flow if the cost incurred on p is minimum among all paths in \mathcal{P}_w . Formally, h^* satisfies

$$\sum_{p \in \mathcal{P}_w} h_p^* = d_w, \forall w \in \mathcal{W}, \text{ and} \quad (1a)$$

$$h_p^* > 0 \implies C_p(h^*) = C_w^{\min}(h^*), \forall p \in \mathcal{P}_w, w \in \mathcal{W}. \quad (1b)$$

Note that if the flow constitutes a WE, then no user can unilaterally reduce their travel cost by choosing a different route. This is due to the underlying assumption that route choice deviation by a single player has no impact on the

aggregate traffic flow. Regarding the existence of WE, in [19, Theorem 5.3], it was established that a WE (i.e., a solution to (1)) exists if for all $p \in \mathcal{P}$, C_p is a positive continuous function. Regarding solvability, most of the approaches for computing a WE rely on complementarity problem formulations. A (nonlinear) *complementarity problem (CP)* is denoted by (X, F) where $X \subseteq \mathbb{R}^n$ and $F : X \rightarrow X$, and the goal is to find a vector $x \in X$ such that $x \geq 0, F(x) \geq 0$ and $x^\top F(x) = 0$. This can be expressed more compactly as

$$0 \leq x \perp F(x) \geq 0. \quad (2)$$

If the map F is affine, then the problem is referred to as a *linear complementarity problem (LCP)*. Following (1), when C_p is a positive function for all $p \in \mathcal{P}$, one can write the WE as the solution to the CP [19] given by

$$0 \leq h_p \perp C_p(h) - v_w \geq 0, \quad \forall w \in \mathcal{W}, p \in \mathcal{P}_w, \quad (3a)$$

$$0 \leq v_w \perp \sum_{p \in \mathcal{P}_w} h_p - d_w \geq 0, \quad \forall w \in \mathcal{W}, \quad (3b)$$

Writing the above CP compactly, as (2), involves considering $x = \begin{pmatrix} h \\ v \end{pmatrix} \in \mathbb{R}_+^{K+|\mathcal{W}|}$ and $F(h, v) := \begin{pmatrix} C(h) - B^\top v \\ Bh - d \end{pmatrix}$, where $C := \{C_p\}_{p \in \mathcal{P}}$ is the vector of cost functions, $B \in \mathbb{R}^{|\mathcal{W}| \times K}$ is the (OD pair, path)-incidence matrix such that $B_{wp} = 1$, if $p \in \mathcal{P}_w$, and 0 otherwise. Note that if (h^*, v^*) is a solution of (3), then h^* is a WE and v_w^* captures the minimum cost among all paths between OD pair w .

III. ROUTING GAME UNDER UNCERTAINTY

The model described thus far was deterministic. However, in real traffic networks the realized costs show significant variability due to factors such as weather, accidents, or traffic signals [12]. Therefore, in this paper, we assume that the cost functions depend on uncertain parameters modeled as random variables. Below, we introduce this stochastic model and discuss existing approaches, along with their limitations, of dealing with the said uncertainties. In the process, we motivate our methodology which follows in the subsequent section. Formally, the uncertain parameters are denoted by u and take values in the set $\mathcal{U} \subseteq \mathbb{R}^L$. The distribution of u is represented by \mathbb{P}_u . The cost functions are now denoted by $C_p : \mathbb{R}_+^K \times \mathcal{U} \rightarrow \mathbb{R}_{++}$. In the presence of uncertainty, the operator F under the CP formulation (2) is given by

$$F(h, v; u) := \begin{pmatrix} C(h; u) - B^\top v \\ Bh - d \end{pmatrix}. \quad (4)$$

Note that for every $u \in \mathcal{U}$, if the cost functions are positive and continuous, then following [19, Theorem 5.3], there exists a WE for every realization of the uncertainty. However, in general, there does not exist a flow that is a WE for all realizations of $u \in \mathcal{U}$, i.e., there is no h^* that satisfies (1) for all $u \in \mathcal{U}$. Consequently, existing approaches either rely on computing a flow that is a solution to a nominal deterministic CP or is a WE for the worst case realization of uncertainty. We briefly summarize these approaches below.

a) *Expected Value Method (EV) [12]*: In this approach, the goal is to solve the expected value complementarity problem given by $0 \leq x \perp \mathbb{E}_{\mathbb{P}_u}[F(x; u)] \geq 0$.

b) *Expected Residual Minimization (ERM)* [10]: Here, the central problem is to $\min_{x \geq 0} \mathbb{E}_{\mathbb{P}_u} [\|\min(x, F(x; u))\|^2]$.

c) *Best Worst Case (BWC)* [12]: The goal here is to solve the CP given by $0 \leq x \perp F^{\text{wc}}(x) \geq 0$, where $F^{\text{wc}}(h, v) = \begin{pmatrix} C^{\text{wc}}(h) - B^T v \\ Bh - d \end{pmatrix}$, with $C_p^{\text{wc}}(h) = \max_{u \in \mathcal{U}} C_p(h; u)$, $\forall p \in \mathcal{P}$. Thus, this approach computes a WE for the worst case travel costs.

d) *Robust Optimization (RO)* [13]: Similar to the BWC framework, here the proposed problem is $\min\{\max_{u \in \mathcal{U}} x^T F(x; u) \mid x \geq 0, F(x; u) \geq 0, \forall u \in \mathcal{U}\}$, which is a robust version of the ERM problem. In [13], it was shown that the above problem is tractable for LCPs.

For both EV and ERM, one needs to know the distribution \mathbb{P}_u , information that is usually unavailable in practice. On the other hand, one does not require \mathbb{P}_u for BWC and RO approaches but the support \mathcal{U} should be known. Furthermore, for BWC and RO, the solution is often conservative and significantly different from the WE for most realizations of the uncertainty, as illustrated in the following example.

Example III.1. Consider the network with two nodes (forming an OD pair) and two arcs. The cost functions on arcs are $C_1(h; u) = h_1$ and $C_2(h; u) = h_2 + u$, where h_1 and h_2 are flows on the arcs. The demand is $d = 100$. The uncertain parameter $u \in [0, 20]$ affects the cost on the second arc. The WE for a fixed $u \in [0, 20]$ is $h_1 = 50 + \frac{u}{2}$ and $h_2 = 50 - \frac{u}{2}$. The BWC approach computes the WE for $u = 20$ and yields $h_1^{\text{bw}} = 60$ and $h_2^{\text{bw}} = 40$. On the other hand, the solution under the RO approach is $h_1^r = h_2^r = 50$ and $v^r = 50$ which corresponds to the WE for $u = 0$. •

In the above example, flows computed under the robust approaches (BWC and RO) are WE for extreme realizations of uncertainty. For all intermediate realizations, the travel costs for the two paths will be very different, i.e., a substantial fraction of users would prefer to deviate from the robust solutions. On the other hand, empirical evidence suggests that users typically do not change their choice of paths based on different realizations of uncertainty (e.g., day-to-day variations of travel time). Instead, they consistently choose paths whose costs are close to the minimum travel cost for most realizations of uncertainty [14]. This phenomenon is modeled in the following section using the notion of regret.

IV. REGRET IN TRAVEL COST UNDER UNCERTAINTY

Consider a traffic flow h and a particular realization of uncertainty $u \in \mathcal{U}$. Consider a user or agent associated with OD pair w who chooses path $p \in \mathcal{P}_w$. Given flow h and uncertainty realization u , the *individual regret* of this agent is defined as the difference between travel cost on path p and the minimum travel cost among all paths in \mathcal{P}_w . Formally, the individual regret is given by $C_p(h; u) - C_w^{\min}(h; u)$. Similarly, given flow h and uncertainty u , the *total regret* is the aggregate of individual regrets, and is stated as

$$R(h; u) := \sum_{w \in \mathcal{W}} \sum_{p \in \mathcal{P}_w} h_p (C_p(h; u) - C_w^{\min}(h; u)). \quad (5)$$

From the definition of WE it follows that if h_u is the WE for the uncertainty realization u , then $R(h_u; u) = 0$. The total regret can also be represented in a compact form as

$$R(h; u) = h^T \tilde{C}(h; u), \quad (6)$$

$$\tilde{C}(h; u) := C(h; u) - B^T C^{\min}(h; u), \quad (7)$$

where $C(h; u) = \{C_p(h; u)\}_{p \in \mathcal{P}}$, $C^{\min}(h; u) = \{C_w^{\min}(h; u)\}_{w \in \mathcal{W}}$, and $B \in \mathbb{R}^{|\mathcal{W}| \times K}$ is the (OD pair, path)-incidence matrix. Note that in the above definition, the minimum travel time can be different for each u , which is in contrast with the CP approach (4) where a dummy variable v is used to capture the minimum travel time.

In the following section, we present data-driven approaches for minimizing, in an appropriate sense, the total regret. The tractability of these approaches rely on the convexity of the regret function in traffic flow. In the remainder of this section, we establish that the function $h \mapsto R(h; u)$ is convex on the set of feasible flows if the travel cost $h \mapsto C(h; u)$ is affine and monotone.¹ The proofs have been omitted due to space reasons and can be reasoned using the fact that the cost is affine and monotone in flow.

Lemma IV.1. *Let $h \mapsto C_p(h; u)$ be an affine function for every $u \in \mathcal{U}$, $p \in \mathcal{P}$. Then, the map $h \mapsto \tilde{C}_p(h; u) := C_p(h; u) - C_w^{\min}(h; u)$ is convex for every given $u \in \mathcal{U}$, $p \in \mathcal{P}_w$, and $w \in \mathcal{W}$.*

We will investigate the convexity of regret among the set of feasible flows satisfying the demand defined as

$$\mathcal{H} := \{h \in \mathbb{R}_+^K \mid \sum_{p \in \mathcal{P}_w} h_p = d_w, \forall w \in \mathcal{W}\}. \quad (8)$$

Lemma IV.2. *For every $u \in \mathcal{U}$, $h \mapsto \tilde{C}(h; u)$ is monotone on the set \mathcal{H} if and only if $h \mapsto C(h; u)$ is monotone on \mathcal{H} .*

We now have the following result as a consequence of Lemmas IV.1 and IV.2. The proof involves checking that $R(tg + (1-t)h) - (tR(g) + (1-t)R(h)) \leq 0$ for all $f, g \in \mathcal{H}$ and $t \in [0, 1]$, and is omitted.

Proposition IV.3. *For every $u \in \mathcal{U}$, the function $h \mapsto R(h; u)$ is convex on the set \mathcal{H} if $h \mapsto C(h; u)$ is affine and monotone on \mathcal{H} .*

The following example shows that \tilde{C} might not be monotone outside \mathcal{H} .

Example IV.4. Consider two paths ($|P| = 2$) between an OD pair, and a monotone cost function $C(h; u) := h$. Accordingly, $\tilde{C}_p(h; u) = h_p - \min_{q \in \{1, 2\}} h_q$. Consider two flows $f = (2, 2)^T$ and $g = (1, 0)^T$ that lead to different total flows or demands. Then $(f-g)^T (\tilde{C}(f; u) - \tilde{C}(g; u)) = -1 < 0$. That is, \tilde{C} is not monotone everywhere. •

V. REGRET MINIMIZATION APPROACHES

The total regret (5) experienced by the users is uncertainty dependent and so, there are different notions of minimizing it. In this section, we present two data-driven approaches

¹A function $F : X \rightarrow \mathbb{R}^n$, $X \subseteq \mathbb{R}^n$ is monotone on X if for every $x, y \in X$, $(x - y)^T (F(x) - F(y)) \geq 0$. Further, F is strictly monotone if the inequality is strict for $x \neq y$.

that minimize the expected value of the regret and a higher quantile of the regret, respectively. The convexity established in Proposition IV.3 plays a key role in ensuring tractability of these data-driven optimization problems.

A. Distributionally Robust Expected Regret Minimization

The problem of interest is

$$\min_{h \in \mathcal{H}} \mathbb{E}_{\mathbb{P}_u} [R(h; u)], \quad (9)$$

where we recall that \mathcal{H} is the set of feasible flows (8). This problem is convex following Proposition IV.3. We assume that \mathbb{P}_u is unknown and instead the decision maker has access to samples or data regarding different realizations of uncertainty. One possible approach of using data is to replace the expectation in (9) by its sample average. However, when the number of samples are too few (and when obtaining more samples is expensive), the solution obtained this way has a higher regret for new samples, that is, it has poor out-of-sample performance [15]. Motivated by these shortcomings, we consider a *distributionally robust* approach where the aim is to minimize a worst-case expected regret computed over a set of distributions (ambiguity set) constructed using data. Formally, let $\widehat{U} = \{\widehat{u}^1, \dots, \widehat{u}^N\} \in \mathcal{U}^N$ be the set of N i.i.d samples drawn from \mathbb{P}_u . The empirical distribution induced by the samples is given by $\widehat{\mathbb{P}}_N := \frac{1}{N} \sum_{i=1}^N \delta_{\widehat{u}^i}$. Consider the ambiguity set $\mathbb{B}_\theta(\widehat{\mathbb{P}}_N) := \{\mathbb{Q} \in \mathcal{M}(\mathcal{U}) \mid d_W(\widehat{\mathbb{P}}_N, \mathbb{Q}) \leq \theta\}$, where $\theta > 0$, $\mathcal{M}(\mathcal{U})$ is the set probability distributions supported on \mathcal{U} having finite first moment, and d_W is the 1-Wasserstein metric (see [15, Definition 3.1] for a formal definition). In particular, $\mathbb{B}_\theta(\widehat{\mathbb{P}}_N)$ contains the set of all distributions that are within a distance θ of the empirical distribution $\widehat{\mathbb{P}}_N$ under the Wasserstein metric. The *distributionally robust expected regret minimization* problem is

$$\min_{h \in \mathcal{H}} \sup_{\mathbb{Q} \in \mathbb{B}_\theta(\widehat{\mathbb{P}}_N)} \mathbb{E}_{\mathbb{Q}} [R(h; u)]. \quad (10)$$

Denote an optimizer and the optimum value of the above problem by \widehat{h}_N and \widehat{v}_N , respectively. Then, following results in [15], we get the performance guarantee

$$\mathbb{P}^N (\mathbb{E}_{\mathbb{P}} [R(\widehat{h}_N; u)] \leq \widehat{v}_N) \geq 1 - \beta,$$

where

$$\beta := \begin{cases} c_1 \exp(-c_2 N \theta^{\max\{2, L\}}), & \text{if } \theta \leq 1, \\ c_1 \exp(-c_2 N \theta^a), & \text{if } \theta > 1, \end{cases}$$

and c_1 , c_2 , and a are positive constants that are independent of the number of samples. The above guarantee implies that given the number of samples, one can vary the radius of the ambiguity set θ to obtain a probabilistic upper bound on the out-of-sample performance of the optimizer of (10). Note that (10) involves optimization over a family of distributions. Using Proposition IV.3 along with [15, Theorem 4.2], we show below that (10) can be equivalently stated as a finite dimensional convex optimization problem, albeit with a large number of constraints. Specifically, define $\mathcal{S}_{\mathcal{P}} := \mathcal{P}_1 \times \mathcal{P}_2 \times \dots \times \mathcal{P}_{|\mathcal{W}|} = \{r^{(1)}, \dots, r^{(|\mathcal{S}_{\mathcal{P}}|)}\}$, where each element in $\mathcal{S}_{\mathcal{P}}$ is a tuple containing one path index for every OD pair. That

is, for any $k \in \{1, \dots, |\mathcal{S}_{\mathcal{P}}|\}$, $r^{(k)} = (r_1^{(k)}, \dots, r_{|\mathcal{W}|}^{(k)})$, where $r_i^{(k)} \in \mathcal{P}_i$ is a path between OD pair i . Given a matrix A with $|\mathcal{P}|$ number of rows, we denote by $A_{r^{(k)}}$ the matrix constituting of rows corresponding to indices in the tuple $r^{(k)}$. The problem (10) can be reformulated as shown below.

Proposition V.1. *Assume that the cost function has the form $C(h) = \tilde{A}u + \tilde{B}h$ where $\tilde{B} \in \mathbb{R}^{K \times K}$ is positive semidefinite and $\tilde{A} \in \mathbb{R}^{K \times L}$. Let the support of uncertainty be $\mathcal{U} = \{u \in \mathbb{R}^L \mid Gu \leq c\}$. Then, the problem (10) is equivalent to the following convex program:*

$$\begin{aligned} & \min_{\substack{h \in \mathcal{H}, \lambda \geq 0, \\ s_i, \gamma_{ik} \geq 0}} \lambda \theta + \frac{1}{N} \sum_{i=1}^N s_i \\ & \text{subject to} \quad \forall k \in \{1, \dots, |\mathcal{S}_{\mathcal{P}}|\}, \forall i \in \{1, \dots, N\} : \\ & \quad b_k(h) + (a_k(h))^\top \widehat{u}^{(i)} + \gamma_{ik}^\top (c - G\widehat{u}^{(i)}) \leq s_i, \\ & \quad \|G^\top \gamma_{ik} - a_k(h)\|_2 \leq \lambda, \end{aligned} \quad (11)$$

where the functions a_k and b_k are defined as follows:

$$a_k(h) = \left(h^\top \tilde{A} - d^\top [\tilde{A}]_{r^{(k)}} \right)^\top, \quad (12)$$

$$b_k(h) = h^\top \tilde{B}h - d^\top [\tilde{B}h]_{r^{(k)}}. \quad (13)$$

The above problem scales poorly to large networks as it has $3|\mathcal{S}_{\mathcal{P}}|$ constraints, where $|\mathcal{S}_{\mathcal{P}}| = \prod_{w \in \mathcal{W}} |\mathcal{P}_w|$, the product of all available paths for each OD pair. Obtaining a more efficient reformulation of (10) with fewer constraints remains as an interesting avenue for future research. Nonetheless, in practice, we can remove paths that are not traversed by users to reduce the number of constraints in (11).

B. Regret Quantile Minimization

Here the aim is to compute a flow that achieves a regret smaller than a given bound for most realizations of uncertainty, i.e., with high probability. This problem can be stated as the following chance constrained optimization

$$\begin{aligned} & \min_{h \in \mathcal{H}, \rho \in \mathbb{R}} \rho \\ & \text{subject to} \quad \mathbb{P}_u [R(h; u) \leq \rho] \geq 1 - \epsilon. \end{aligned} \quad (14)$$

For the following discussion we assume that R is convex in h for every fixed $u \in \mathcal{U}$. The problem (14) computes a flow, and the smallest bound such that the regret for this flow is smaller than the bound with high probability. In other words, we minimize the $1 - \epsilon$ quantile of the regret R . Solving (14) is often computationally intractable. We assume that \mathbb{P}_u is unknown and N i.i.d samples $\{\widehat{u}^1, \dots, \widehat{u}^N\}$ drawn from \mathbb{P}_u are available. We compute an approximate solution of (14) via the scenario approach where the constraint $R(h; u) \leq \rho$ is required to hold for all available samples. This leads to the following convex optimization problem

$$\min \{ \rho \in \mathbb{R} \mid h \in \mathcal{H}, R(h; \widehat{u}^i) \leq \rho, \forall i \in \{1, \dots, N\} \} \quad (15)$$

Scenario approach comes with the guarantee that if N is sufficiently large, then the optimal solution of (15), denoted by (h_N, ρ_N) , satisfies the probabilistic constraint in (14) with high probability. Specifically, if $N \geq \frac{2}{\epsilon} \left(K + \ln \left(\frac{1}{\beta} \right) \right)$,

Method	Expected regret	Expected regret std	Flow distance
EV	72652.835	-	6.657
RO	144534.440	-	11.542
BWC	971413.995	-	51.219
RS N=50	80585.979	10083.543	7.335
RS N=100	77213.246	6556.573	7.001
RS N=500	88048.749	14495.041	7.546
DRER N=50 $\theta=0.01$	70771.304	1034.154	6.834
DRER N=100 $\theta=0.01$	70144.706	460.606	6.737
DRER N=500 $\theta=0.01$	69783.021	98.970	6.723

TABLE I: Results for five-link network with Beta distribution

where $\beta \in (0, 1)$ and K is the total number of paths. Then, following [18], we have

$$\mathbb{P}_u^N(\mathbb{P}_u(R(h_N; u) \leq \rho_N) \geq 1 - \epsilon) \geq 1 - \beta. \quad (16)$$

We now evaluate the performance of both data-driven methods via simulations.

VI. SIMULATIONS

We compare using simulations the proposed data-driven regret minimization approaches to the existing expectation based and robust solutions discussed in Section III. We consider two networks: a two node five link network and a simplified Sioux Falls network. We assume that the travel cost is affine in the traffic flow and the uncertain parameters, and the demand between different OD pairs is constant. We set $\mathcal{U} = [0, 1]^M$ for some positive integer M . The samples are i.i.d and the distribution is either uniform on $[0, 1]$ or Beta with parameters $\alpha = 2$ and $\beta = 10$.

For the distributionally robust expected regret (DRER) minimization approach, we solve the problem (11) with 50, 100, 500, and 1000 samples and for each sample size with the Wasserstein radii $\theta \in \{0.1, 0.01, 0.001, 0.0001\}$. We also solve the scenario program (15) with 50, 100, 500 and 1000 samples, and refer the resulting approach as the regret scenario (RS) approach. We compare the flows obtained via these methods to the ones obtained by the expected value method (EV), best worst case (BWC) approach and the robust optimization (RO), see Section III for their respective definitions. We compute the flows via DRER approach only for the smaller two node five link network as the program (11) is not currently scalable to problems of larger sizes.

Each data-driven program was run 25 times with different samples to obtain 25 different traffic flow vectors. We then compute the distribution of regret for the resulting flows empirically using 100000 new samples of the uncertainty. We also used 10000 new samples to get a sample average approximation for the expected arc flow distance $\mathbb{E}_{\mathbb{P}_u}[\|h - h^{WE}(u)\|_2]$, where h is the solution arc flow and $h^{WE}(u)$ is the WE arc flow for a given realization of uncertainty u . The arc-flow distance serves as another metric to compare the solutions obtained by different methods. We summarize the results obtained for different networks below.

1) *Five-Link Network*: We first consider a network with two nodes $\mathcal{N} = \{A, B\}$, and with three links $\{1, 2, 3\}$ going from A to B and two links $\{4, 5\}$ going from B to A . The available paths are the set of all links and they are ordered as

$(1, 2, 3, 4, 5)$ in all vectors. The network and cost functions are adapted from [13, Section 6.3]. The demand is 260 from A to B , and is 170 from B to A . The vector of cost functions is given by

$$C(h; u) = \begin{pmatrix} 40h_1 + 20h_4 + 1000 + 3730.967u_1 \\ 60h_2 + 20h_5 + 950 \\ 80h_3 + 3000 \\ 8h_1 + 80h_4 + 1000 + 4696.115u_2 \\ 4h_2 + 100h_5 + 1300 \end{pmatrix}. \quad (17)$$

Table I shows the expected regret (mean and standard deviation over the 25 different runs), and empirical expected arc flow distance for the flows computed by the different methods when uncertainty is sampled from a Beta distribution. DRER has the lowest expected regret, followed by the EV and RS methods. Both robust approaches have significantly higher mean regrets and flow distances. Figure 1a shows the expected regret for DRER under different sample sizes and Wasserstein radii. Note that N and θ do not have a large influence on the expected regret compared to the RS approach where the expected regret increases significantly as the number of samples increases. Figures 1b and 1c show the average (over 25 independent runs) empirical CDF of the regret for uniform and Beta distributions, respectively. The average empirical CDFs are very similar for all flows obtained by the DRER, RS and EV techniques. Flows obtained via the robust optimization and BWC methods have higher regret quantiles for almost all probabilities.

2) *Simplified Sioux Falls Network*: The Sioux Falls dataset is a well-studied traffic network of medium size (24 nodes and 76 arcs), and is available at the transportation networks for research repository [20]. We consider the travel time function (widely used Bureau of Public Roads (BPR) function [21]) for an arc $a \in \mathcal{A}$ as $l_a(f_a) = t_a \left(1 + b_a \left(\frac{f_a}{c_a}\right)\right)$, and the cost of a path is the sum of the costs of its arcs. We set $b_a = 100$ and for the free flow travel time t_a , capacity c_a for arc a , and demand between OD pairs, we use the values given by the dataset. We used only three OD pairs $\mathcal{W} = \{(1, 19), (13, 8), (12, 18)\}$ and choose a subset of 10 paths that have the shortest free flow time for each OD pair. The simplified instance reduces computational burden and helps us better interpret the results.

We introduced uncertainty in the travel time as before. We added uncertainty only to the subset of the 18 arcs that contain at least one of the nodes 10, 16 and 17 by adding $2t_a u_a$ to their travel time. As discussed in Ordóñez and Stier-Moses [12], these nodes represent the downtown area. Figures 1d and 1e show the average empirical CDF for uniform and Beta distributions, respectively. In both cases, the RS method has better regret quantiles for probabilities above 0.4. The CDF of the BWC method is not shown because it is significantly large. The flows and the CDFs obtained by the RS method do not vary significantly as the number of samples increases. In Figure 1f, we compare the CDFs of the regret obtained via the RS method for different sample sizes when the probabilities are close to 1. In this regime, the flows obtained by considering a larger number

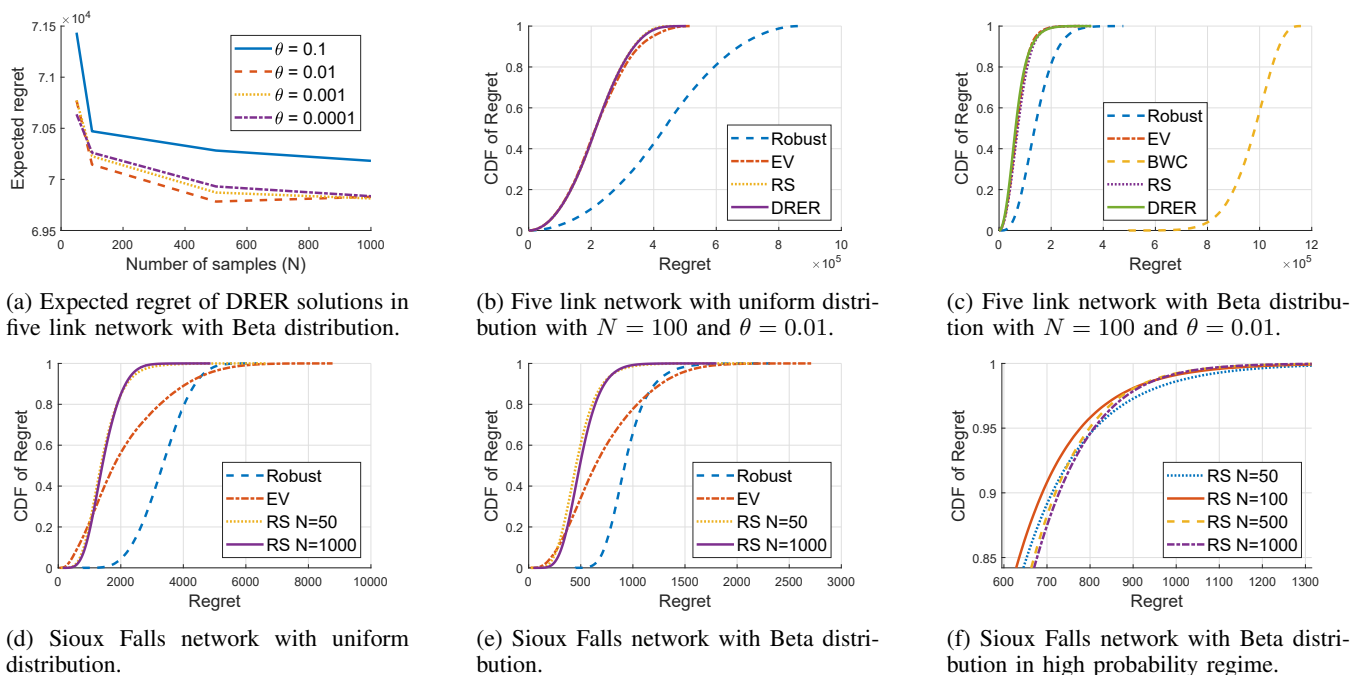


Fig. 1: Average expected regret and average empirical CDF of regret obtained by different methods.

of samples have smaller regret.

VII. CONCLUSIONS

We consider routing games under uncertainty, and formally define the notion of regret. We prove that when costs are affine in the traffic flow, the aggregate regret is convex in the flow for each realization of uncertainty. We then proposed two data-driven methods: one, to minimize expected regret in a distributionally robust manner, and the other, to minimize higher quantiles of regret via scenario optimization. We compared the flows obtained by our methods to several robust and expectation-based approaches via extensive simulations.

In future, we plan to design tractable methods to (approximately) minimize regret when costs are non-affine and when the demand is uncertain and potentially dependent on the traffic flow. Another important aspect is to characterize the equilibrium when agents choose paths to minimize either the expected value or a quantile of their individual regret.

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