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Explicit simplicial discretization of distributed-parameter port-Hamiltonian systems

Marko Seslija a,1, Jacquelin M.A. Scherpen b, Arjan van der Schaft c

a Kasteelpark Arenberg 10, B-3001 Leuven, Belgium
b Institute for Technology, Engineering and Management, University of Groningen, Nijenborgh 4, 9747 AG Groningen, The Netherlands
c Johann Bernoulli Institute for Mathematics and Computer Science, University of Groningen, Nijenborgh 9, 9747 AG Groningen, The Netherlands

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A B S T R A C T

Simplicial Dirac structures as finite analogues of the canonical Stokes–Dirac structure, capturing the topological laws of the system, are defined on simplicial manifolds in terms of primal and dual cochains related by the coboundary operators. These finite-dimensional Dirac structures offer a framework for the formulation of standard input–output finite-dimensional port-Hamiltonian systems that emulate the behavior of distributed-parameter port-Hamiltonian systems. This paper elaborates on the matrix representations of simplicial Dirac structures and the resulting port-Hamiltonian systems on simplicial manifolds. Employing these representations, we consider the existence of structural invariants and demonstrate how they pertain to the energy shaping of port-Hamiltonian systems on simplicial manifolds.

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1. Introduction

A wide class of field theories can be treated as port-Hamiltonian systems (Schöberl & Schlacher, 2011; van der Schaft & Maschke, 2002). The Stokes–Dirac structure defined by van der Schaft and Maschke (2002) is an infinite-dimensional Dirac structure which provides a theoretical account that permits the inclusion of varying boundary variables in the boundary problem for partial differential equations. From an interconnection and control viewpoint, such a treatment of boundary conditions is essential for the incorporation of energy exchange through the boundary, since in many applications the interconnection with the environment takes place precisely through the boundary. For numerical integration, simulation and control synthesis, it is of paramount interest to have finite-dimensional approximations that can be interconnected to one another.

Most of the numerical techniques emanating from the field of numerical analysis, however, fail to capture the intrinsic system structures and properties, such as symplecticity, conservation of momenta and energy, as well as differential gauge symmetry. Mixed finite element methods can be constructed in such a manner that a number of important structural properties are preserved (Bossvait, 1998; Hiptmair, 2002; Hirani, 2003). Most of the efforts have been focused on systems on manifolds without boundary or zero energy flow through the boundary. In Golo, Talasila, van der Schaft, and Maschke (2004) a mixed finite element scheme for structure-preserving discretization of port-Hamiltonian systems was proposed. The construction is clear in a one-dimensional spatial domain, but becomes complicated for higher spatial domains. Furthermore, the geometric content of the discretized variables remains moot, in sense that, for instance, the boundary variables do not genuinely live on the geometric boundary.

Recently in Seslija, van der Schaft, and Scherpen (2012), we suggested a discrete exterior geometry approach to structure-preserving discretization of distributed-parameter port-Hamiltonian systems. The spatial domain in the continuous theory represented by a finite-dimensional smooth manifold is replaced by a homological manifold–like simplicial complex and its circumcircular dual. The smooth differential forms, in discrete setting, are mirrored by cochains on the primal and dual complexes, while the discrete exterior derivative is defined to be the coboundary operator. Discrete analogues of the Stokes–Dirac structure are the so-called simplicial Dirac structures defined on spaces of primal and dual discrete differential forms. These finite-dimensional
Dirac structures offer a natural framework for the formulation of finite-dimensional port-Hamiltonian systems that emulate their infinite-dimensional counterparts. The resulting port-Hamiltonian systems are in the standard input–output form, unlike in Golo et al. (2004), where the discretized models are acausal (given by a set of differential and algebraic equations). The explicit input–output form obtained by our scheme has the advantage from both numerical and control perspective over the implicit model presented in Golo et al. (2004).

In this paper, we address the issue of matrix representations of simplistic Dirac structures by representing cochains by their coefficient vectors. In this manner, all linear operator from the continuous world can be represented by matrices, including the Hodge star, the coboundary and the trace operator. Firstly, we recall the definition of the Stokes–Dirac structure and port- Hamiltonian systems. In the third section, we define some essential concepts from discrete exterior calculus as developed by Desbrun, Hirani, Leok, and Marsden (2002) and Hirani (2003). In order to allow the inclusion of nonzero boundary conditions on the dual cell complex, in Seslija et al. (2012) we have adapted a definition of the dual boundary operator that leads to a discrete analogue of the dual boundary operator that lead to a discrete analogue of the integration-by-parts formula, which is a crucial ingredient in establishing simplistic Dirac structures on a primal simplicial complex and its circumcentric dual. We demonstrate how these simplistic Dirac structures relate to the spatially discretized wave equation on a bounded domain and to the telegraph equations on a segment. Towards the end of the paper, we consider the existence of structural invariants, which are crucial for the control by energy shaping.

Goal and contributions. This paper is written with several purposes in mind.

- The essential theoretical results of this paper pertaining to structure-preserving discretization, namely, Sections 3–5 have been already reported in Seslija, Scherpen, and van der Schaft (2011, 2012); Seslija et al. (2012) in an algebraic topology setting. The results in this paper do not lean onto the heavy nomenclature of algebraic topology, but instead emphasize matrix representations, making it more accessible and easier to implement. We demonstrate that a discrete differential modeling approach to consistent discretization of distributed-parameter systems is quite approachable—and, in fact, is often much simpler than its continuous counterpart.

- We aim to render the theoretic foundation of our exposition accessible to control theorists, and the paper as such serves as a segue to the rich literature on the subject.

- Another contribution of this paper is given in Sections 7 and 8. Here we address the existence of dynamical invariants for the obtained spatially discrete systems and look at the energy-Casimir method for energy shaping. We anticipate that this line of research will lead to more elaborate and fruitful control strategies for distributed systems.

- We hope that by the end of the paper it will become clear that the discrete geometry-based approach to modeling is not only tied to the discretization of infinite-dimensional systems, but, instead, stands as a potent language for the system and control community.

2. Background of port-Hamiltonian systems

Dirac structures were originally developed by Courant (1990) and Dorfman (1993) as a generalization of symplectic, presymplectic and Poisson structures. Later, Dirac structures were employed as the geometric formalism underpinning generalized interconnected and constrained Hamiltonian systems (van der Schaft, 2000; van der Schaft & Maschke, 2002).

2.1. Dirac structures

Let X be a manifold and define a pairing on TX ⊕ T⁎X given by $$\langle \langle f_1, e_1 \rangle, \langle f_2, e_2 \rangle \rangle = \langle e_1|f_2 \rangle + \langle e_2|f_1 \rangle.$$ For a subspace D of TX ⊕ T⁎X, we define the orthogonal complement D⊥ as the space of all (f₁, e₁) such that $$\langle \langle f_1, e_1 \rangle, \langle f_2, e_2 \rangle \rangle = 0$$ for all (f₂, e₂). A Dirac structure is then a subbundle D of TX ⊕ T⁎X which satisfies D D⊥.

The notion of Dirac structures is suitable for the formulation of closed Hamiltonian systems, however, our aim is a treatment of open Hamiltonian systems in such a way that some of the external variables remain free port variables. For that reason, let F be a linear vector space of external flows, with the dual space F⁎ of external efforts. We deal with Dirac structures on the product space X × F. The pairing on (TX × F) ⊕ (T⁎X × F⁎) is given by $$\left\langle \left( \langle f_1, f_2 \rangle, \langle e_1, e_2 \rangle \right) \right\rangle = \langle e_1|f_2 \rangle + \langle e_2|f_1 \rangle.$$ (1)

A generalized Dirac structure D is a subbundle of (TX × F) ⊕ (T⁎X × F⁎) which is maximally isotropic under (1).

Consider a generalized Dirac structure D on the product space X × F. Let H : X → ℝ be a Hamiltonian. The port-Hamiltonian system corresponding to a 4-tuple (X, F, D, H) is defined by a set of smooth time-functions [t ↦ (x(t), fᵢ(t), eᵢ(t)) ∈ X × F × F⁎] for t ∈ [0, T] satisfying the equation

$$\dot{x}(t) = fᵢ(t), \quad \dot{H}(x(t), eᵢ(t)) \in D \quad \text{for } t \in I.$$ (2)

The Eq. (2) implies the energy balance $$\frac{d}{dt}[H(x(t))] = \langle \partial H(x(t))|\dot{x}(t) \rangle.$$ An important class of finite-dimensional port-Hamiltonian systems is given by

$$\dot{x} = J(\alpha)(\partial H \partial x)(x) + g(x)eᵢ$$

$$fᵢ = g'(\alpha)(\partial H \partial x)$$

where for clarity we have omitted the argument t, and J : T⁎X → TX is a skew-symmetric vector bundle map and g : F → TX is the independent input vector field.

In this work, we deal exclusively with Dirac structures on linear spaces, which can be defined as follows. Let F and E be linear spaces. Given an f ∈ F and an e ∈ E, the pairing will be denoted by (e|f) ∈ ℝ. By symmetrizing the pairing, we obtain a symmetric bilinear form (e|f) : F × E → ℝ naturally given as $$\langle \langle (e_1|f_1), (e_2|f_2) \rangle \rangle = \langle (e_1|f_2) + (e_2|f_1) \rangle.$$ A constant Dirac structure is a linear subspace D ⊂ F × E such that D D⊥, with ⊥ standing for the orthogonal complement with respect to the bilinear form (e|f).

2.2. Stokes–Dirac structure

The Stokes–Dirac structure is an infinite-dimensional Dirac structure that provides a foundation for the port-Hamiltonian formulation of a class of distributed-parameter systems with boundary energy flow (van der Schaft & Maschke, 2002).

Hereafter, let M be an oriented n-dimensional smooth manifold with a smooth (n − 1)-dimensional boundary ∂M endowed with the induced orientation, representing the space of spatial variables. Adhering to the familiar ground in this paper, M shall be a bounded Euclidean domain. By $$\Omega^k(M), k = 0, 1, \ldots, n$$, denote the space of exterior k-forms on M, and by $$\Omega^k(\partial M), k = 0, 1, \ldots, n − 1$$, the space of k-forms on ∂M.
For any pair $p, q$ of positive integers satisfying $p + q = n + 1$, define the flow and effort linear spaces by

$$\mathcal{F}_{p,q} = \Omega^p(M) \times \Omega^q(M) \times \Omega^{n-p-q}(\partial M)$$
$$\mathcal{E}_{p,q} = \Omega^{n-p}(M) \times \Omega^{n-q}(M) \times \Omega^{n-p-q}(\partial M).$$

The bilinear form on the product space $\mathcal{F}_{p,q} \times \mathcal{E}_{p,q}$ is

$$\langle (f^1, f^1_\nu f^2_\nu f^3_\nu, e^1, e^1\nu e^1_\nu e^1_\nu e^1_\nu e^1_\nu, e^2_\nu e^2_\nu e^2_\nu e^2_\nu e^2_\nu), (f^3, f^3_\nu f^2_\nu f^1_\nu, e^3, e^3\nu e^3_\nu e^3_\nu e^3_\nu e^3_\nu, e^1_\nu e^2_\nu e^3_\nu e^3_\nu e^3_\nu) \rangle_{\mathcal{F}_{p,q} \times \mathcal{E}_{p,q}}$$

$$= \int_M e^1_\nu f^1_\nu + e^1 f^2_\nu + e^1_\nu f^3_\nu + e^2 f^1_\nu + e^2_\nu f^1_\nu + e^3 f^1_\nu$$

$$+ \int_{\partial M} e^1_\nu f^2_\nu + e^2_\nu f^1_\nu.$$  \hspace{1cm} (4)

**Theorem 1** (van der Schaft & Maschke, 2002). Given linear spaces $\mathcal{F}_{p,q}$ and $\mathcal{E}_{p,q}$, and the bilinear form $\langle \cdot, \cdot \rangle$, define the following linear subspace $\mathcal{D}$ of $\mathcal{F}_{p,q} \times \mathcal{E}_{p,q}$

$$\mathcal{D} = \{(f_p, f_q, f_b, e_p, e_b, \nu) \in \mathcal{F}_{p,q} \times \mathcal{E}_{p,q} | \nu \in \partial M, \nu \in \partial M \}$$

\[
\begin{bmatrix}
    f_p \\
    f_q \\
    f_b \\
    e_p \\
    e_b \\
    \nu
\end{bmatrix} =
\begin{bmatrix}
    0 & (-1)^{p+1}d & e_p \\
    d & 0 & e_q \\
    \text{tr} & 0 & (-1)^n \text{tr} \\
\end{bmatrix}
\]

$$\text{where } d \text{ is the exterior derivative and } \text{tr stands for the trace operator on the boundary } \partial M. \text{ Then } \mathcal{D} = \mathcal{D}^\perp, \text{ that is, } \mathcal{D} \text{ is a Dirac structure.}$$

Consider a Hamiltonian density $\mathcal{H} : \Omega^p(M) \times \Omega^q(M) \rightarrow \Omega^q(M)$ resulting in the Hamiltonian $\mathcal{H} = \int_M \mathcal{H} \in \mathbb{R}$. Setting the flows $f_p = -\frac{\partial \mathcal{H}}{\partial e_p} f_q = -\frac{\partial \mathcal{H}}{\partial e_q}$ and the efforts $e_p = \delta_p H, e_q = \delta_q H$, where $(\delta_p H, \delta_q H) \in \Omega^{n-p}(M) \times \Omega^{n-q}(M)$ are the variational derivatives of $H$ at $(\alpha_p, \alpha_q)$, the distributed-parameter port-Hamiltonian system is defined by the relation

$$\left( -\frac{\partial e_p}{\partial t} - \frac{\partial e_q}{\partial t}, f_b, \delta_p H, \delta_q H, e_b \right) \in \mathcal{D}, \hspace{1cm} t \in \mathbb{R}.$$

Since $di = \int_{\partial M} e_b \wedge f_b$, the system is lossless.

### 3. Basics of discrete exterior calculus

In the discrete setting, the smooth manifold $M$ is replaced by an oriented manifold-like simplicial complex. An $n$-dimensional simplicial manifold $K$ is a simplicial triangulation of an $n$-dimensional polytope $|K|$ with an $(n - 1)$-dimensional boundary. Familiar examples of such discrete manifolds are meshes of triangles embedded in $\mathbb{R}^3$ and tetrahedra obtained by tetrahedralization of 3-dimensional manifolds.

#### 3.1. Chains and cochains

The discrete analogue of a smooth $k$-form on the manifold $M$ is a $k$-cochain on the simplicial complex $K$. A $k$-chain is a formal sum of $k$-simplices of $K$ such that its value on a simplex changes sign when the simplex orientation is reversed. The free Abelian group generated by a basis consisting of oriented $k$-simplices with real-valued coefficients is $\mathcal{C}_k(K; \mathbb{R})$. The space $\mathcal{C}_k(K; \mathbb{R})$ is a vector space with dimension equal to the number of $k$-simplices in $K$, which is denoted by $N_k$. The space of $k$-cochains is the vector space dual of $\mathcal{C}_k(K; \mathbb{R})$ denoted by $\mathcal{C}^k(K; \mathbb{R})$ or $\mathcal{H}^k(K)$, as a reminder that this is the space of discrete $k$-forms.

The discrete exterior derivative or the coboundary operator $d_k : \mathcal{H}^k(K) \rightarrow \mathcal{H}^{k+1}(K)$ is defined by duality to the boundary operator $\partial_{k+1} : \mathcal{C}^{k+1}(K; \mathbb{Z}) \rightarrow \mathcal{C}_k(K; \mathbb{Z})$, with respect to the natural pairing between discrete forms and chains. For a discrete form $\alpha \in \mathcal{H}^k(K)$ and a chain $c_{k+1} \in \mathcal{C}^{k+1}(K; \mathbb{Z})$ we define $d_k$ by

$$d_k \alpha, c_{k+1} = \langle \alpha, d^k \rangle c_{k+1} = \langle \alpha, \partial_{k+1} c_{k+1} \rangle,$$

where the boundary operator $\partial_{k+1}$ is the incidence matrix from the space of $(k + 1)$-simplices to the space of $k$-simplices and is represented by a sparse $N_k \times N_{k+1}$ matrix containing only 0 or ±1 elements (Desbrun, Kanso, & Tong, 2008). The important property of the boundary operator is $\partial_{k+1} \circ \partial_{k+1} = 0$. The exterior derivative also satisfies $d^k \circ d^k = 0 = d^k \circ d^{k+1}$, which is a discrete analogue of the vector calculus identities curl $\circ$ grad = 0 and div $\circ$ curl = 0.

**Remark 1.** The relation (7) can be regarded as a discrete Stokes’ theorem, where the role of the exterior derivative is being played by the coboundary operator and the discrete analogue of integration is the evaluation of a cochain.

### 3.2. Dual cell complex

An essential ingredient of discrete exterior calculus is the dual complex of a manifold-like simplicial complex. The main idea is to associate to each primal $k$-simplex a dual $(n - k)$-cell. For example, in the 3-dimensional case, consider a tetrahedral mesh with interior elements shown in Fig. 1. We associate a dual 3-cell to each primal vertex (0-simplex), a dual polygon (2-cell) to each primal edge (1-simplex), a dual edge (1-cell) to each primal face (2-simplex), and a dual vertex (0-cell) to each primal tetrahedron (3-simplex).

In the 2-dimensional case, for illustration consider the triangular mesh in Fig. 2. To the primal edge $[v_i, v_j]$ we associate the dual edge $[\hat{v}_i, \hat{v}_j]$, where the vertices $\hat{v}_i$ and $\hat{v}_j$ are the circumcenters of the two neighboring triangles that share the common edge $[v_i, v_j]$. The dual edge of $[v_i, v_j]$ will be denoted by $\star_1[v_i, v_j]$. The dual of the vertex $v_i$ is its Voronoi region shown shaded. The dual of the face $[v_{in}, v_{ip}, v_{in}]$ is its circumcenter $\hat{v}_i$, while the dual of the edge $[v_{in}, v_{ip}]$ is the (half-)edge $[\hat{v}_i, \hat{v}_i] = \star_1[v_{in}, v_{ip}]$. The dual of the edge $\star_1[v_i, v_j]$ is its cover relative to $|K|$.

---

\[2\] The circumcenter of a $k$-simplex $\sigma^k$ is given by the center of the $k$-circumsphere, which is the unique $k$-sphere that has all $k + 1$ vertices of $\sigma^k$ on its surface.
The just explained geometric duality is the so-called circum-centric or Voronoi duality,\footnote{In algebraic topology \cite[Munkres, 1984]{Munkres} and computational electromagnetics \cite[Bossavit, 1998; Hiptmair, 2002]{Bossavit, Hiptmair}, another popular choice of the geometric dualism is barycentric duality.} which has an important property that primal and dual cells are orthogonal to each other. This feature dramatically simplifies the discrete counterpart of the Hodge star, as will be shown in the next subsection. For this reason we shall be dealing with the circum-centric duality and require that the simplicial complex is well-centered (the circumcenters of all simplices of all dimensions lie in the interior of the corresponding simplices).

Given a simplicial complex $K$, we define its interior dual cell complex $\star K$ as the circumcentric dual of $K$ geometrically restricted to $|K|$.

In a similar fashion, to each primal $k$-simplex living on the geometric boundary of $K$, hereafter denoted by $\partial K$, we can uniquely associate an $(n - 1 - k)$-cell living on the dual of the boundary $\partial \hat{K}$. The circumcentric dual of $\partial K$ is the boundary dual cell complex $\star_b K$. For example, considering Fig. 2, on the boundary, the dual of the edge $[v_2, v_3]$ is the dual vertex $\hat{v_3}$, while the boundary dual of the primal vertex $v_3$ is the curvilinear edge $\hat{v_3}$, shown bolded.

The dual cell complex $\star K$ is defined as $\star K = \star_b K \times \star_\circ K$. The dual mesh $\star K$ is a dual to $K$ in sense of a graph dual, and the dual of the boundary is equal to the boundary of the dual, that is $\partial (\star K) = \star (\partial K) = \star_b K$. Because of duality, there is: (1) a one-to-one correspondence between $k$-simplices of $K$ and interior $(n - k)$-cells of $\partial \hat{K}$; (2) a bijection between primal $k$-simplices of $\partial K$ and the dual boundary $(n - 1 - k)$-cells of $\star_b K$.

### 3.3. Exterior derivatives on the dual mesh

Everything that has been said about the primal discrete forms carries over to the dual cochains, which can be interpreted as covectors. The space of dual $k$-cochains will be denoted as $\Omega^k_d(\star K)$. The covectors will be labeled by a caret symbol, e.g., $\hat{v} \in \Omega^1_d(\star K)$.

The trace operator $\text{tr}^1: \Omega^2_d(K) \to \Omega^1_d(\partial K)$ is a matrix that isolates the members of a $k$-cochain vector assumed on the geometric boundary $\partial K$.

The dual exterior derivative $d_d^{\circ - k}: \Omega^n_d(\star K) \to \Omega^{n-k+1}_d(\star K)$ is defined by duality to the primal exterior operator $d$ as

$$d_d^{\circ - k} = (-1)^k (d^{-1})^t.$$  

The negative sign appears as the orientation of the dual is induced by the primal orientation.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{fig2}
\caption{A 2-dimensional simplicial complex $K$ and its circumcentric dual cell complex $\star K$ indicated by dashed lines. The boundary of $\star K$ is the dual of the boundary of $K$.}
\end{figure}

\begin{figure}
\centering
\includegraphics[width=\textwidth]{fig3}
\caption{The simplicial complex $K$ consisting of two triangles. The dual edges introduced by subdivision are shown dotted. The shaded area is the dual cell $\hat{v}_1$ of the primal vertex $v_1$.}
\end{figure}

The dual boundary exterior derivative $d_d^{\circ - k}: \Omega^n_d(\star K) \to \Omega^{n-k+1}_d(\star K)$ is defined as

$$d_d^{\circ - k} = (-1)^k (\text{tr}^{n-k+1})^t.$$  

\begin{example}
Consider a simplicial complex pictorially given by Fig. 3. The primal and dual 2-faces have counterclockwise orientations. The matrix representation of the incidence operator $\partial_1$, from the primal edges to the primal vertices, is

$$\begin{bmatrix}
v_0 & v_1 & v_2 & v_3 & v_4
\end{bmatrix}$$

while the discrete exterior derivative from the vertices to the edges is the transpose of the incidence operator, i.e., $d_1^0 = \partial_1^t$. The dual exterior derivative is $d_1^t = - (d_1^0)^t$, while the matrix representation of the $d_1^t$ operator is

$$\begin{bmatrix}
\hat{v}_2 & \hat{v}_3 & \hat{v}_4 & \hat{v}_5 & \hat{v}_6
\end{bmatrix} \begin{bmatrix}
\hat{v}_0 & \hat{v}_1 & \hat{v}_2 & \hat{v}_3 & \hat{v}_4
\end{bmatrix}$$

Here, as previously explained, $\hat{v}_i, v_i$ is the Voronoi dual of a vertex $v_i$ and $[\hat{v}_2, \hat{v}_3], [\hat{v}_1, \hat{v}_3], [\hat{v}_3, \hat{v}_4], [\hat{v}_4, \hat{v}_5]$ are the boundary duals of $v_0, v_1, v_2, v_3$, respectively.

The trace operator is $\text{tr}^1 = (d_1^t)^t$.

The incidence operator $\partial_2$, from the set of primal face to the set of the primal edges, is

$$\begin{bmatrix}
\hat{v}_0 & \hat{v}_1 & \hat{v}_2 & \hat{v}_3 & \hat{v}_4
\end{bmatrix} \begin{bmatrix}
\hat{v}_0 & \hat{v}_1 & \hat{v}_2 & \hat{v}_3 & \hat{v}_4
\end{bmatrix}$$

The $d_1^t$ operator is

$$\begin{bmatrix}
\hat{v}_1 & \hat{v}_2 & \hat{v}_3 & \hat{v}_4
\end{bmatrix} \begin{bmatrix}
\hat{v}_0 & \hat{v}_1 & \hat{v}_2 & \hat{v}_3 & \hat{v}_4
\end{bmatrix}$$

The trace operator that isolates the elements living on the boundary edges is $\text{tr}^1 = - (d_1^t)^t$. 

\end{example}
3.4. Discrete wedge and Hodge operator

There exists a natural pairing, via the so-called primal–dual wedge product, between a primal \( k \)-cochain and a dual \( (n - k) \)-cochain. Let \( \alpha \in \Omega^k_d(K) \) and \( \beta \in \Omega^{n-k}_d(*K) \). We define the discrete primal–dual wedge product \( \wedge : \Omega^k_d(K) \times \Omega^{n-k}_d(*K) \to \mathbb{R} \) by

\[
\langle \alpha \wedge \beta \rangle = \sum_{\sigma^k \in K} (\alpha|_{\sigma^k}) \langle \beta |_{\sigma^k}, \sigma^k \rangle = \alpha^t \beta
\]

where the summation is taken over all \( k \)-simplices \( \sigma^k \) in \( K \). Analogously, for an \( \alpha \in \Omega^k_d(K) \) and a \( \beta \in \Omega^{n-k}_d(*K) \), we define the primal–dual pairing on the boundary by

\[
\langle \alpha_b \wedge \beta_b \rangle, \partial K = \alpha^t_b \beta - (1)^{k(n-k)} \langle \beta \wedge \alpha, \partial K \rangle.
\]

**Remark 2.** Given a primal \((k - 1)\)-form \( \alpha \) and an internal dual \((n - k)\)-discrete form \( \beta \in \Omega^{n-k}_d(*K) \) and a dual boundary form \( \beta_b \in \Omega^{n-k}_d(*K) \), then

\[
\langle d^{k-1} \alpha \wedge \beta_b, K \rangle + (1)^{k-1}(\alpha \wedge (d^{k-1} \beta + \beta_b^{k-1} \beta_b), K) = (d^{k-1} \alpha \wedge \beta_b, \partial K).
\]

The last relation is the *summation-by-parts formula* and its validity is an immediate consequence of \( d^{k-1} \alpha = (-)^k (d^{k-1} \alpha)^t \) and \( d^{k-1} = (-)^{k-1} (d^{k-1})^t \).

The support volumes of a simplex and its dual cell are the same, which suggests that there is a natural identification between primal \( k \)-cochains and dual \((n - k)\)-cochains. In the exterior calculus for smooth manifolds, the Hodge star, denoted \( *_k \), is an isomorphism between the space of \( k \)-forms and \((n - k)\)-forms. The *discrete Hodge star* is a map \( *_k : \Omega^k_d(K) \to \Omega^{n-k}_d(*K) \) defined by its value over simplices and their duals. In the case of the circumcentric duality, the Hodge star \( *_k \) is a diagonal \( N_k \times N_k \) matrix with the entry corresponding to a simplex \( \sigma^k \) being \( |\sigma^k| \times |\sigma^k| \), that is

\[
*_k = \text{diag} \left( \frac{|\sigma^1|}{|\sigma^1|}, \frac{|\sigma^2|}{|\sigma^2|}, \ldots, \frac{|\sigma^N_k|}{|\sigma^N_k|} \right).
\]

Here, \( |\sigma^k| \) and \( |\sigma^k| \) are the volumes of \( \sigma^k \) and \( \sigma^k \), respectively.

**Example 2.** Consider the 2-dimensional simplicial complex and its circumcentric dual in Fig. 3. The diagonal Hodge operators are

\[
*_0 = \text{diag} \left( |\ast_0|, |\ast_1|, |\ast_2| \right),
\]

\[
*_1 = \text{diag} \left( \left\langle [v_0, v_1], [\hat{v}_0, \hat{v}_2], [\hat{v}_2, \hat{v}_0] \right\rangle, \left\langle [v_0, v_1], [\hat{v}_1, \hat{v}_3], [\hat{v}_3, \hat{v}_1] \right\rangle, \left\langle [v_1, v_2], [\hat{v}_1, \hat{v}_3], [\hat{v}_3, \hat{v}_1] \right\rangle \right) \text{ and}
\]

\[
*_2 = \text{diag} \left( \left\langle [v_0, v_1], [v_0, v_2], [v_1, v_2] \right\rangle \right).
\]

**Remark 3.** Another possibility for the construction of the Hodge operator is to use Whitney forms. The Whitney map is an interpolation scheme for cochains. It maps discrete forms to square integrable forms that are piecewise smooth on each simplex. The Whitney maps are built from barycentric coordinate functions and the resulting matrix is sparse but in general not diagonal (Bossvait, 1998; Hiptmair, 2002).

The linear operators of discrete exterior calculus used in this paper are succinctly presented in the following diagram

\[
\Omega^0_d(K) \xrightarrow{\partial^0} \Omega^1_d(K) \xrightarrow{*_0} \Omega^n_d(*K) \xrightarrow{d^{n-1}} \Omega^{n-1}_d(*K)
\]

\[
\Omega^1_d(K) \xrightarrow{\partial^1} \Omega^2_d(K) \xrightarrow{*_1} \Omega^{n-1}_d(*K) \xrightarrow{d^{n-2}} \Omega^{n-2}_d(*K)
\]

\[
\vdots
\]

\[
\Omega^{n-1}_d(*K) \xrightarrow{\partial^{n-1}} \Omega^0_d(K) \xrightarrow{*_n} \Omega^n_d(*K)
\]

4. Simplicial Dirac structures

In this section, we develop the matrix representations of simplicial Dirac structures. These structures are discrete analogues of the Stokes–Dirac structure and as such are defined in terms of primal and duals cochains on the underlying discrete manifold.

In the discrete setting, the role of the bounded domain \( M \) is played by an \( n \)-dimensional well-centered oriented manifold-like simplicial complex \( K \). The flow and the effort spaces will be the spaces of complementary primal and dual forms. The elements of these two spaces are paired via the discrete primal–dual wedge product. Let

\[
\mathcal{F}^d_{p,q} = \Omega^p_d(K) \times \Omega^{n-q}_d(*K) \xrightarrow{\partial^q} \Omega^p_{d,q-1}(K)
\]

\[
\mathcal{E}^d_{p,q} = \Omega^p_d(K) \times \Omega^{n-q}_d(*K) \xrightarrow{\partial^q} \Omega^p_{d,q-1}(K)
\]

The primal–dual wedge product ensures a bijection between the primal and dual forms, between the flows and efforts. A natural discrete mirror of the bilinear form (4) is a symmetric pairing on the product space \( \mathcal{F}^d_{p,q} \times \mathcal{E}^d_{p,q} \) defined by

\[
\langle \langle f^p, f^q, \beta^p, \beta^q \rangle, \langle \hat{f}^p, \hat{f}^q, \hat{\beta}^p, \hat{\beta}^q \rangle \rangle_d = \langle e^p, e^q, e^p, e^q \rangle_d \]

\[
= \langle e^p \wedge f^q + \hat{e}^q \wedge f^p + e^q \wedge \hat{f}^p + \hat{e}^p \wedge f^q, K \rangle
\]

\[
= \langle e^p \wedge f^q + \hat{e}^q \wedge f^p + e^q \wedge \hat{f}^p + \hat{e}^p \wedge f^q, \partial K \rangle.
\]

A discrete analogue of the Stokes–Dirac structure is the finite-dimensional Dirac structure constructed in the following theorem.

**Theorem 2.** Given linear spaces \( \mathcal{F}^d_{p,q} \) and \( \mathcal{E}^d_{p,q} \) and the bilinear form \( \langle \cdot, \cdot \rangle_d \). The linear subspace \( \mathcal{D}_d \subset \mathcal{F}^d_{p,q} \times \mathcal{E}^d_{p,q} \) defined by

\[
\mathcal{D}_d = \left\{ \langle \hat{f}, f, \beta, \hat{\beta} \rangle \in \mathcal{F}^d_{p,q} \times \mathcal{E}^d_{p,q} \right\}
\]

\[
\left[ \begin{array}{c}
\hat{f}_p \\
\hat{e}_p
\end{array} \right] = \left[ \begin{array}{c}
(1)^r d^{1-q} \hat{e}_p \\
0
\end{array} \right] \left[ \begin{array}{c}
e^q \\
0 \end{array} \right] + \left[ \begin{array}{c}
(1)^r d^{1-q} \\
0 \end{array} \right] \hat{e}_p,
\]

\[
f_p = (1)^r d^{1-q} \hat{e}_p.
\]

with \( r = pq + 1 \), is a Dirac structure with respect to the pairing \( \langle \cdot, \cdot \rangle_d \).
Proof. Note that since $d_i^{n-q} = (-1)^q(d^{n-p})^q$ and $d_i^{n-q} = (-1)^{n-p}$ $(tr^{n-p})$, the operator
\[
\begin{bmatrix}
0 & (-1)^q d_i^{n-q} & (-1)^q d_i^{n-q} \\
(-1)^q tr^{n-p} & 0 & 0 \\
(-1)^q tr^{n-p} & 0 & 0
\end{bmatrix}
\]
is skew-symmetric, and thus (9) is a Poisson structure on the state space $\Omega_d^d(\ast, K) \times \Omega_d^d(K)$. □

The other discrete analogue of the Stokes–Dirac structure is defined on the spaces
\[
\hat{f}_{p,q}^d = \Omega_d^p(K) \times \Omega_d^q(\ast, K) \rightarrow \Omega_d^{n-q}(\ast, K) \times \Omega_d^{n-p}(K) \times \Omega_d^{n-q}(\ast, K) \\
\hat{e}_{p,q}^d = \Omega_d^{n-q}(\ast, K) \times \Omega_d^{n-q}(\ast, K) \times \Omega_d(n-p)(\ast, K) \times \Omega_d(n-p)(\ast, K)
\]
A natural discrete mirror of (4) in this case is a symmetric pairing defined by
\[
\langle\langle (f_p^d, f_q^d, f_b^d, \hat{e}_p^d, \hat{e}_q^d, \hat{e}_b^d), (f_p^d, f_q^d, f_b^d, \hat{e}_p^d, \hat{e}_q^d, \hat{e}_b^d) \rangle\rangle = \delta_p \delta_q \left( \epsilon_p \hat{e}_p + \epsilon_q \hat{e}_q + \epsilon_b \hat{e}_b \right)
\]
\[
\epsilon_p \hat{e}_p + \epsilon_q \hat{e}_q + \epsilon_b \hat{e}_b = (-1)^q d_i^{n-q} e_i + (-1)^q d_j^{n-q} e_j
\]
is a Dirac structure with respect to the bilinear pairing $\langle\langle \cdot, \cdot \rangle\rangle$.

Proof. The simplicial Dirac structure (10) is the dual of (9), and the proof is analogous to that of Theorem 2. □

In the following section, the simplicial Dirac structures (9) and (10) will be used as terminus a quo for the geometric formulation of spatially discrete port-Hamiltonian systems.

5. Port-Hamiltonian systems

Let a function $H : \Omega_d^d(\ast, K) \times \Omega_d^d(K) \rightarrow \mathbb{R}$ stand for the Hamiltonian $(\alpha_p, \alpha_q) \mapsto H(\alpha_p, \alpha_q)$, with $\hat{\epsilon}_p \in \Omega_d^d(K)$ and $\alpha_q \in \Omega_d^d(K)$. A time derivative of $H$ along an arbitrary trajectory $t \mapsto (\alpha_p(t), \alpha_q(t)) \in \Omega_d^d(\ast, K) \times \Omega_d^d(K)$, $t \in \mathbb{R}$, is
\[
\frac{d}{dt} H(\alpha_p, \alpha_q) = \frac{\partial H}{\partial \alpha_p} \partial \alpha_p + \frac{\partial H}{\partial \alpha_q} \partial \alpha_q + \frac{\partial \alpha_p}{\partial \alpha_q} \partial \alpha_q + \frac{\partial \alpha_q}{\partial \alpha_q} \partial \alpha_q,
\]
where the caret sign reminds that the quantity lives on the dual mesh. The relations between the simplicial-Dirac structure (9) and time derivatives of the variables are: $f_p = -\hat{\epsilon}_p \hat{\alpha}_p$, $f_q = -\hat{\epsilon}_q \hat{\alpha}_q$, while the efforts are: $e_p = \frac{\partial H}{\partial \alpha_p}$, $\hat{e}_q = \frac{\partial H}{\partial \alpha_q}$.

This allows us to define a time-continuous port-Hamiltonian system on a simplicial complex $K$ (and its dual $\ast K$) by
\[
\begin{bmatrix}
0 & (-1)^q d_i^{n-q} & (-1)^q d_i^{n-q} \\
(-1)^q tr^{n-p} & 0 & 0 \\
(-1)^q tr^{n-p} & 0 & 0
\end{bmatrix}
\]

The system (12) is evidently in the form (3). It immediately follows that $\frac{d}{dt} \mathcal{H} = \mathcal{E}_b \cdot \mathcal{F}_b$, enunciating a fundamental property of the system: the increase in the energy on the domain $|K|$ is equal to the power supplied to the system through the boundary $\partial K$ and $\partial (\ast K)$. The boundary efforts $\mathcal{E}_b$ are the boundary control input and $\mathcal{F}_b$ are the outputs.

Remark 4. Introducing a linear negative feedback control as $\mathcal{E}_b = (-1)^{n-p}(q \cdot q^{1-p} - q)\mathcal{F}_b$, where $\mathcal{F}_b$ is the Hodge star on the boundary $\partial K$, leads to passivization of the lossless port-Hamiltonian system, i.e., $\frac{d}{dt} \mathcal{H} \leq -(\mathcal{F}_b \cdot \mathcal{E}_b, F_b) \leq 0$. Furthermore, if the Hamiltonian is a $K_{\infty}$ function with a strict minimum that is a stationary set for the system (12), the equilibrium is asymptotically stable. A more elaborate control strategy can be the energy shaping method as is briefly discussed in Section 8.

An alternative formulation of a spatially discrete port-Hamiltonian system is given in terms of the simplicial Dirac structure (10). We start with the Hamiltonian function $(\alpha_p, \alpha_q) \mapsto H(\alpha_p, \alpha_q)$, where $\alpha_p \in \Omega_d^d(K)$ and $\alpha_q \in \Omega_d^d(K)$. In a similar manner as in deriving (12), we introduce the input–output port-Hamiltonian system
\[
\begin{bmatrix}
\frac{\partial \mathcal{E}_p}{\partial t} \\
\frac{\partial \mathcal{E}_q}{\partial t} \\
\frac{\partial \mathcal{F}_b}{\partial t}
\end{bmatrix} = \begin{bmatrix}
0 & (-1)^q d_i^{n-q} & 0 \\
(-1)^q d_i^{n-q} & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
\frac{\partial \mathcal{H}}{\partial \alpha_p} \\
\frac{\partial \mathcal{H}}{\partial \alpha_q} \\
\frac{\partial \mathcal{H}}{\partial \alpha_b}
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix} \mathcal{F}_b,
\]
\[
e_b = (-1)^q tr^{n-q} \frac{\partial \mathcal{H}}{\partial \alpha_q}
\]
In contrast to (12), in the case of the formulation (13), the boundary flows $\mathcal{F}_b$ can be considered to be freely chosen, while the boundary efforts $\mathcal{E}_b$ are determined by the dynamics. Note that the free boundary variables are always defined on the boundary of the dual cell complex.

6. Physical examples

In this section, we consider the discrete wave equation on a 2-dimensional simplicial complex and the telegraph equations on a segment.

6.1. Two-dimensional wave equation

Consider the wave equation $\mu \frac{d^2 u}{dt^2} = -\Delta u$, with $u(t, z) \in \mathbb{R}, z = (z_1, z_2) \in \mathbb{M}$, where $\mu$ is the mass density, $E$ is the Young’s modulus, $\Delta$ is the two-dimensional Laplace operator, and $\mathbb{M}$ is a compact surface with a closed boundary. Throughout, the superscript $c$ designates the continuous quantities.

The energy variables are the 2-dimensional kinetic momentum $p^c$, and the 1-form elastic strain $e^c$. The coenergy variables are the 0-form velocity $v^c$ and the 1-form stress $\sigma^c$. The energy density of the vibrating membrane is $H(p, e) = \frac{1}{2}(e^c \cdot \sigma^c + p^c \cdot v^c)$, where the coenergy and energy variables are related by the constitutive relations $\sigma^c = E \cdot e^c$ and $v^c = 1/\mu \cdot p^c$. The Hodge operator here corresponds to the standard Euclidean metric on $\mathbb{M}$. The port-Hamiltonian formulation of the vibrating membrane in full details is given in Golo et al. (2004).

Let us now consider the simplicial Dirac structure underpinning the discretized two-dimensional wave equation. The energy variables of the discretized system are chosen as follows: the kinetic momentum is a dual 2-form whose time derivative is set to be $f_b$, the elastic strain is a primal 1-form with time derivative corresponding to $f_b$, the coenergy variables are a primal 0-form $e_p$ and a dual 1-form $e_q$. Such a formulation of the discrete wave
The primal 1-dimensional simplicial complex \( K \). By construction, the nodes \( v_0 \) and \( v_{2n} \) are added to the boundary to insure that \( \partial (\delta K) = \ast (\delta K) \).

The coenergy variables are the 1-form stress \( \hat{e}_b \), while the output is the boundary velocity. The Hamiltonian of the discrete model is

\[
\mathcal{H} = \frac{1}{2} \left( \varepsilon \wedge E \ast_1 \varepsilon + \hat{p} \wedge \frac{1}{\mu} \ast_0^{-1} \hat{p}, K \right).
\]

The coenergy variables are the dual 1-form \( \hat{q} = \frac{\delta q}{\delta t} = \ast_0 \hat{p} \) and the primal 0-form \( v = \frac{\delta v}{\delta t} = \hat{v}_b \).

The resulting port-Hamiltonian system is

\[
\begin{bmatrix}
\frac{\partial \hat{p}}{\partial t} & \frac{\partial \hat{q}}{\partial t}
\end{bmatrix} = \begin{bmatrix}
\hat{d}_0^1 & e^0
\end{bmatrix} \begin{bmatrix}
0 & 1
\end{bmatrix} \begin{bmatrix}
\hat{p}_0^1 & 0
\end{bmatrix} \hat{e}_b,
\]

\[
f_b = \frac{1}{\mu} \hat{e}_b.
\]

where the operators \( \hat{d}_0^1, d_0^1, \hat{d}_0^0, d_0^0 \) conform to the diagram at the end of Section 3 when \( n = 2 \).

**Example 3.** For the Dirac structure (14) on the simplicial manifold \( K \) given in Fig. 3, the operators \( d_0^1, d_0^0, \hat{d}_0^0 \) are given in Example 1. It is straightforward to show

\[
\begin{align*}
\partial (d_0^0 \varepsilon) & \wedge \hat{e}_b, K) + (\hat{e}_b \wedge \partial (d_0^0 e_b) + d_0^1 \hat{e}_b, K) = \hat{e}_b [v_0, v_1] f_n(b_0) + \hat{e}_b [v_0, v_2] f_n(b_1) + \hat{e}_b [v_0, v_3] f_n(b_2),
\end{align*}
\]

what confirms that the boundary terms genuinely live on the boundary of \( |K| \).

### 6.2. Telegraph equations

We consider an ideal lossless transmission line on a 1-dimensional simplicial complex given in Fig. 4. The energy variables are the charge density \( q = \Omega_2^0(K) \), and the flux density \( \hat{q} = \Omega_1^1(K) \), hence \( p = q = 1 \). The Hamiltonian representing the total energy stored in the transmission line with distributed capacitance \( C \) and distributed inductance \( L \) is

\[
\mathcal{H} = \frac{1}{2C} q \wedge \ast_1 q + \frac{1}{2L} \hat{q} \wedge \ast_0^{-1} \hat{q}, K \tag{16}
\]

where \( \ast_0 \) and \( \ast_1 \) are the discrete diagonal Hodge operators that relate the appropriate cochains according to the following schematic diagram

\[
\begin{align*}
\Omega_0^0(\partial K) \xrightarrow{\hat{d}_0^0} & \Omega_2^0(K) \xrightarrow{\hat{d}_0^1} \Omega_1^1(K) \xrightarrow{\ast_0} \Omega_0^0(K) \xrightarrow{\hat{d}_0^0} \Omega_2^0(\partial K) \\
\downarrow \ast_0 & \downarrow \ast_1 \quad \downarrow \ast_0 & \downarrow \ast_1
\end{align*}
\]

The physical realization of this is that, in contrast to our results, the transmission line in the finite-dimensional case is not only composed of inductors and capacitors but also of transformers.

**Example 5.** The discrete analogue of the Stokes–Dirac structure obtained in Gol’d et al. (2004) is a finite-dimensional Dirac structure, but not a Poisson structure. The implication of this on the physical realization is that. Stabilization of either of those systems is easily achieved by terminating boundary ports with resistive elements, what is a practical application of the passivization explained in Remark 4.

**Remark 6.** The accuracy of the proposed method is \( 1/n \) (see Seslija et al., 2012).

### 7. Conservation laws

Let us consider the existence of conservation laws and structural invariants for the port-Hamiltonian systems on simplicial complexes.
7.1. Finite-dimensional invariants

The following proposition gives the conditions for the existence of conservation laws in the discrete setting.

**Proposition 4.** Consider the port-Hamiltonian system (12). Let \((\hat{\alpha}, \hat{\rho}, \phi, \epsilon) \mapsto C(\hat{\alpha}, \hat{\rho}, \phi, \epsilon)\) be a real-valued function. Then

\[
\frac{\partial C}{\partial \hat{\alpha}} \in \ker d^{n-p} \quad \text{and} \quad \frac{\partial C}{\partial \hat{\rho}} \in \ker d^{1-p},
\]

iff \(C\) is a conservation law for the port-Hamiltonian system (12) satisfying

\[
\frac{dc}{dt} = (f^{(1)})^\tau \hat{e}_b,
\]

where \(f^{(1)} = (\infty^{(1)+1})\) is given by (13), \(\hat{e}_b\) is the vector of states.

**Proof.** Differentiating \(C\) along the flow of the system (12), we have

\[
\begin{align*}
\frac{dc}{dt} &= \frac{\partial C}{\partial \hat{\alpha}} \frac{d\hat{\alpha}}{dt} + \frac{\partial C}{\partial \hat{\rho}} \frac{d\hat{\rho}}{dt} + \frac{\partial C}{\partial \phi} \frac{d\phi}{dt} + \frac{\partial C}{\partial \epsilon} \frac{d\epsilon}{dt} + K \\
&= (\infty^{(1)}) \left( d^{n-p} \frac{\partial C}{\partial \hat{\alpha}} \frac{d\hat{\alpha}}{dt} + d^{1-p} \frac{\partial C}{\partial \hat{\rho}} \frac{d\hat{\rho}}{dt} \right) \\
&\quad - (\infty^{(1)n-q}) \left( d^{n-q} \frac{\partial C}{\partial \phi} \frac{d\phi}{dt} + d^{q} \frac{\partial C}{\partial \epsilon} \frac{d\epsilon}{dt} \right) \\
&\quad - (\infty^{(1)+p+1}) \left( d^{p} \frac{\partial C}{\partial \hat{\alpha}} \frac{d\hat{\alpha}}{dt} \right) + \left( d^{1-p} \frac{\partial C}{\partial \hat{\rho}} \frac{d\hat{\rho}}{dt} \right) \frac{d\hat{e}_b}{dt} \\
&\quad + (\infty^{(1)q}) \left( d^{q} \frac{\partial C}{\partial \phi} \frac{d\phi}{dt} \right) \frac{d\hat{e}_b}{dt} + (\infty^{(1)q}) \left( d^{q} \frac{\partial C}{\partial \epsilon} \frac{d\epsilon}{dt} \right) \frac{d\hat{e}_b}{dt} + K
\end{align*}
\]

where we have used the fact that \(d^{n-p} = (\infty^{(1)}) \) and \(d^{1-q} = (\infty^{(1)+1}) \). Furthermore, regardless of \(H\), the result (23) follows iff (21) and (22) hold.

\[\Box\]

**Remark 7.** If either \(\hat{e}_b = 0\) or \(f^{(1)} = 0\), the quantity \(C\) satisfying (21) and (22) is a conserved quantity—a Casimir function.

7.2. One-dimensional domain

An interesting case for which it is possible explicitly to solve (21) is when \(p = n\). The matrix \(d^{1-q}\) is nothing but the transpose of the incidence matrix \(\hat{\alpha}\), from the set of edges to the set of vertices, on a connected graph. It is a well known property of any incidence matrix \(\hat{\alpha}\) that \(\ker \hat{\alpha} = \text{span} \mathbf{1}\), where \(\mathbf{1}\) stands for the vector with all elements equal 1. A direct consequence of this is that \(\frac{dc}{dt} = 1\) up to a multiplicative constant.

In the one-dimensional case the null space of \(d^{1-q}\) is trivial, cf. (19), which allows us to explicitly express the conservation law.

**Corollary 5.** Consider the port-Hamiltonian system (12), with \(p = n\), on a one-dimensional simplicial manifold given on Fig. 4. The quantity \(C \in \mathbf{1}^\tau \hat{\alpha} = \hat{\alpha}_p(\{V_0, V_1\}) + \sum_{k=1}^{n-1} \hat{\alpha}_p(\{V_{2k-1}, V_{2k}\}) + \hat{\alpha}_p(\{V_{2n-1}, V_{2n}\})\) satisfies the balance law

\[
\frac{dc}{dt} = \hat{e}_b \frac{d\phi}{dt} - \hat{e}_b \frac{d\epsilon}{dt}.
\]

In case of the telegraph equations on the segment \(M = [0, 1]\), the total charge \(c = \int_0^1 \phi^\tau (t, z) dz\) as well as the total magnetic flux \(c = \int_0^1 \phi^\tau (t, z) dz\) are both conservation laws. In the discrete setting, the only conservation law for the system (18) is the total charge \(c = \int 1^\tau \hat{\alpha}_p\) whose derivative along the admissible trajectories is \(\frac{dc}{dt} = \hat{e}_b \frac{d\phi}{dt} - \hat{e}_b \frac{d\epsilon}{dt}\). Similarly, the total flux \(c = \int 1^\tau \phi\) in the system (17) satisfies the balance law \(\frac{dc}{dt} = \hat{e}_b \frac{d\phi}{dt} - \hat{e}_b \frac{d\epsilon}{dt}\), where \(\hat{e}_b\) and \(\hat{e}_b\) are input currents. These result differ from those presented in Macchelli (2011), where both the total flux and total charge are conserved.

8. Energy-Casimir method

Consider the interconnection of (12) with the (possibly nonlinear) integrator

\[
\begin{align*}
\frac{dx}{dt} &= g_x \frac{dH_x}{d_x} \\
y_x &= g_x \frac{dH_x}{d_x}
\end{align*}
\]

where \(\xi \in \mathbb{R}^n\), \(\xi \in \mathbb{R}^m\), \(N = \dim \Omega_2^{n-q}(\hat{\alpha}(\rho, \xi))\), input \(u_x\), output \(y_x\), and \(\xi \mapsto H_x(\xi)\) the controller’s Hamiltonian. The interconnection is power-preserving with \(u_x = f_x\) and \(e_x = -y_x\). The composition is the port-Hamiltonian system in the form

\[
\begin{bmatrix}
\hat{\alpha}_p \\
\hat{\alpha}_q \\
\hat{\rho} \\
\hat{\phi} \\
\hat{\epsilon} \\
\hat{H}_x
\end{bmatrix}
\frac{dt}{dt}
\begin{bmatrix}
\hat{\alpha}_p \\
\hat{\alpha}_q \\
\hat{\rho} \\
\hat{\phi} \\
\hat{\epsilon} \\
\hat{H}_x
\end{bmatrix}
= \begin{bmatrix}
0 \\
-d^{n-p} \frac{d^{1-q} \hat{H}_x}{d_x} + \frac{d^{1-q} \hat{H}_x}{d_x} \\
-d^{n-p} \frac{d^{1-q} \hat{H}_x}{d_x} + \frac{d^{1-q} \hat{H}_x}{d_x} \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
\frac{d^{1-q} \hat{H}_x}{d_x} \\
\frac{d^{1-q} \hat{H}_x}{d_x} \\
\frac{d^{1-q} \hat{H}_x}{d_x} \\
\frac{d^{1-q} \hat{H}_x}{d_x} \\
\frac{d^{1-q} \hat{H}_x}{d_x} \\
\frac{d^{1-q} \hat{H}_x}{d_x}
\end{bmatrix}
\]

with \((\hat{\alpha}_p, \hat{\alpha}_q, \xi) \mapsto H_x(\hat{\alpha}_p, \hat{\alpha}_q, \xi)\) is the closed-loop Hamiltonian \(H_x(\hat{\alpha}_p, \hat{\alpha}_q, \xi) = \hat{H}(\hat{\alpha}_p, \hat{\alpha}_q) + H_x(\xi)\).

The energy shaping for the system (28) is achieved by restricting the behavior of (28) to a certain subspace (van der Schaft, 2000). To this end, we look at the Casimir functions of the closed-loop system.
Proposition 6. The real-valued function \((\hat{\alpha}_p, \alpha_\eta, \zeta) \mapsto C(\hat{\alpha}_p, \alpha_\eta, \zeta)\) is a Casimir function of the closed system (28) iff
\[
\begin{bmatrix}
\frac{\partial C}{\partial \hat{\alpha}_p} \\
\frac{\partial C}{\partial \alpha_\eta} \\
\frac{\partial C}{\partial \zeta}
\end{bmatrix} \in \text{ker } d^{m-p} \cap \text{ker } (g, tr^{m-p})
\]

\[
\frac{\partial C}{\partial \alpha_\eta} = \text{ker } d^i - (-1)^{m-i} d^{n-m} g_i.
\]  

(29)

Proof. Solving \(\frac{\partial}{\partial t} C(\hat{\alpha}_p, \alpha_\eta, \zeta) = 0\) irrespective of \(H_d\) directly leads to (29).

Remark 8. Since the structural matrix of the port-Hamiltonian system (12) is not of full rank in case when \(g_i\) is identity, not all Casimirs of (28) are of the form \(C(\hat{\alpha}_p, \alpha_\eta, \zeta) = S(\hat{\alpha}_p, \alpha_\eta) - \zeta_i\), \(i = 1, \ldots, m\).

Remark 9. In case when \(p = q = m = n = 1\) and \(g_i = [1, 1]\), the only Casimir for the system (28) is \(1/2 \alpha_\eta + \zeta\).

9. Final remarks

The explicit simplicial discretization treated in this paper leads to the standard input–output port-Hamiltonian systems without algebraic constraints. The analysis and the control synthesis for such systems belong to the realm of standard finite-dimensional systems.

In the last section, we looked at a simple control strategy for the energy shaping of discretized port-Hamiltonian systems. This attempt has only scratched the surface of a very important problem. Since the discretized model assumes a port-Hamiltonian structure, much more elaborate schemes for the control of port-Hamiltonian systems can be applied. A nontrivial problem in this regard would be to design a controller for the discretized model and then test it on the continuous model and obtain the bounds of the discrepancy norm between the two behaviors. Some initial work has already been done in this vein, however, mostly pertaining to the systems on a one-dimensional spatial domain (see Macchelli, 2011; Voss & Scherpen, 2011 and references quoted therein). In higher dimensions, the interconnection of the finite controller and the infinite-dimensional plant would be naturally realized through the interface of the simplicial triangulation of the boundary. Gauging the input–output errors and energy shaping of the closed-loop systems are ideas we plan to explore in forthcoming publications.

References


Jacquelin M.A. Scherpen received the M.Sc. and Ph.D. degrees in Applied Mathematics from the University of Twente, Enschede, The Netherlands, in 1988 and 1994, respectively. She was with the Delft University of Technology, The Netherlands, from 1994 to 2006. Since September 2006, she has been a professor with the University of Groningen, Institute for Industrial Technology and Management of the Faculty of Mathematics and Natural Sciences, The Netherlands, where she was recently appointed as a scientific director. She has held visiting research positions at the University of Tokyo, Japan, Université de Compiegne, and SUPÉLEC, Gif-sur-Yvette, France, and the Old Dominion University, Norfolk, VA, USA. Her current research interests include nonlinear model reduction methods, nonlinear control methods, modeling and control of physical systems with applications to electrical circuits, electromechanical systems and mechanical systems, and distributed optimal control applications to smart grids. She has been an Associate Editor of the IEEE Transactions on Automatic Control and the International Journal of Robust and Nonlinear Control (IJRNC). Currently, she is an Associate Editor of the IMA Journal of Mathematical Control and Information. She is on the editorial board of the IJRCN.

Marko Seslija is a Postdoctoral Scholar at the Stadius Center for Dynamical Systems, Signal Processing and Data Analytics of TU Leuven. He received his Ph.D. degree from the Faculty of Mathematics and Natural Sciences, the University of Groningen, The Netherlands, in March 2013. Employing geometric compositional modeling strategies and computational algebraic topology, he strives to enhance the analysis and control of complex systems stemming from engineering and systems biology.

A.J. (Arjan) van der Schaft received the undergraduate and Ph.D. degrees in Mathematics from the University of Groningen, The Netherlands. In 1982 he joined the Department of Applied Mathematics, University of Twente, Enschede, where he was appointed as a Full Professor in Mathematical Systems and Control Theory in 2000. In September 2005 he returned to his Alma Mater as a Full Professor in Mathematics. He has served as an Associate Editor for Systems & Control Letters, Journal of Nonlinear Science, SIAM Journal on Control and Optimization, and the IEEE Transactions on Automatic Control. Currently he is Associate Editor for the IEEE Transactions on Automatic Control. Arjan van der Schaft received a Diploma in Electrical and Electronics Engineering (IEEE). He was an Invited Speaker at the International Congress of Mathematicians, Madrid, 2006. He was the recipient of the three-yearly awarded ‘Certificate of Excellent Achievements from the IFAC Technical Committee on Nonlinear Control Systems’ in 2013. His research interests include mathematical systems theory, nonlinear systems and control theory, systems modeling and analysis of multi-domain physical systems, geometric mechanics and hybrid systems. His present focus is on network modeling and analysis of complex physical systems and their geometric formulation as port-Hamiltonian systems, and their applications in power systems, cyber-physical systems, and systems biology.