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A Simple Approach to Distributed Observer Design for Linear Systems

Weixin Han^{ID}, Harry L. Trentelman^{ID}, *Fellow, IEEE*, Zhenhua Wang^{ID}, and Yi Shen^{ID}, *Member, IEEE*

Abstract—This note investigates the distributed estimation problem for continuous-time linear time-invariant (LTI) systems observed by a network of observers. Our starting point is a given LTI system whose full state vector we want to estimate, based on the measured output of the system. We will call this the observed system. Each observer in the network has access to only part of the output of the observed system, and communicates with its neighbors according to a given network graph. In this note, we recover the known result that if the observed system is observable and the network graph is a strongly connected digraph, then a distributed observer exists. Moreover, the estimation error can be made to converge to zero at any *a priori* given decay rate. Our approach leads to a relatively straightforward proof of this result, using the mirror of the balanced graph associated with the original network graph. The numerical design of our distributed observer is reduced to solving linear matrix inequalities. Each observer in the network has state dimension equal to that of the observed plant.

Index Terms—Distributed estimation, linear matrix inequalities (LMIs), linear system observers, sensor networks.

I. INTRODUCTION

In recent years, distributed state estimation has received a lot of attention in response to increasing demand for estimating the state of a dynamic system over spatially deployed multiagents or networked sensors [1]. A linear dynamical system is monitored by a network of sensor nodes. The objective of each node is to asymptotically estimate the state of the dynamical system using its own (limited) measurements and via information exchange with neighbors. This is known as the distributed state estimation problem [2]. The main challenge in the distributed estimation problem comes from the limitation that no single observer can estimate the state of the system only using its local measurement. Thus, classical observer design methods cannot be directly applied to the distributed estimation problem [3].

A similar distributed estimation problem for linear dynamical systems has been studied using various approaches. These approaches can be classified into two main branches, namely: Kalman-filter-based techniques, and observer-based techniques. Distributed and decentralised Kalman filters have been studied in [4]–[7], where the observed system

Fig. 1. Framework for distributed state estimation [14].

is separated into several interconnected subsystems. There, each local estimator has reduced order and only estimates a part of the state vector. A distributed Kalman-filter approach based on an average-consensus concept to estimate the global state was proposed in [8] and [9]. These methods rely on a two-step strategy: a Kalman-filter-based state estimate update rule and a data fusion step based on average-consensus [10]. A single-time-scale strategy and design a scalar-gain estimator was proposed in [11] and [12]. Sufficient conditions for stability of the estimator are given in their work. However, the tight coupling between the network topology and the plant dynamics limits the range of the scalar gain parameter.

On the other hand, an observer-based approach is studied under the joint observability assumption. In [3], [13], and [14], a state augmentation observer is constructed to cast the distributed estimation problem as the problem of designing a decentralized stabilizing controller for an LTI plant, using the notion of fixed modes [15]. It should be noted that these works only discuss discrete-time systems. In [16], a general form of distributed observer with arbitrarily fast convergence rate was proposed. Based on the Kalman observable canonical decomposition, local Luenberger observers at each node are constructed in [2], [17], and [18]. The observer reconstructs a certain portion of the state solely by using its own measurements, and uses consensus dynamics to estimate the unobservable portions of the state at each node. Specifically, in [1] two observer gains are designed to achieve distributed state estimation, one for local measurements and the other for the information exchange.

In this note, we study the distributed observer problem for continuous-time LTI systems observed by a network of Luenberger observers (see Fig. 1 for an illustration). Each observer accesses a portion of the output of the observed LTI system, and communicates with its neighboring observers. The observer at each node is designed to asymptotically estimate the full state of the observed system. Unlike the state augmentation observer approach in [3], [13], and [14] and also the observer construction in [16], in this note, the local Luenberger observer at each node has the same order as the observed system. We borrow the idea of observability decomposition from [1]. The estimation error of the i th observer in the observable part is stabilized by output injection, and the estimation in the unobservable part reaches a consensus with the other local observer's estimates. Compared with [1], in this note the information among the local observers is exchanged

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W. Han, Z. Wang, and Y. Shen are with the Department of Control Science and Engineering, Harbin Institute of Technology, Harbin 150001, China (e-mail: hanweixin2009@gmail.com; zhenhua.wang@hit.edu.cn; yishen_hit@126.com).

H. L. Trentelman is with the Johann Bernoulli Institute for Mathematics and Computer Science, University of Groningen, Groningen 9700 AK, The Netherlands (e-mail: h.l.trentelman@rug.nl).

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by a strongly connected directed graph. Furthermore, our algorithmic procedure to compute a suitable distributed observer is different from [1] and is based on solving LMIs, which makes it attractive numerically. Also, our algorithm allows any desired error decay rate. We relax the design constraints and develop a new simple design procedure. The choice of the gain matrices in the distributed observer becomes more flexible. We decouple the topological information of the network from the local gain matrices of the observer by the introduction of an auxiliary undirected graph. This auxiliary graph is obtained by balancing the original graph and then taking the mirror of this balanced graph, see [19]. The gain matrices in the observer can be obtained by solving linear matrix inequalities (LMIs), which makes the distributed observer design numerically feasible.

II. PRELIMINARIES AND PROBLEM FORMULATION

A. Preliminaries

Notation: For a given matrix M , its transpose is denoted by M^T and M^{-1} denotes its inverse. The symmetric part of a square real matrix M is sometimes denoted by $\text{Sym}(M) := M + M^T$. The rank of the matrix M is denoted by $\text{rank } M$. The identity matrix of dimension N will be denoted by I_N . The vector $\mathbf{1}_N$ denotes the $N \times 1$ column vector comprising of all ones. For a symmetric matrix P , $P > 0$ ($P < 0$) means that P is positive (negative) definite. For a set $\{A_1, A_2, \dots, A_N\}$ of matrices, we use $\text{diag}\{A_1, A_2, \dots, A_N\}$ to denote the block diagonal matrix with A_i s along the diagonal, and the matrix $[A_1^T \ A_2^T \ \dots \ A_N^T]^T$ is denoted by $\text{col}(A_1, A_2, \dots, A_N)$. The Kronecker product of the matrices M_1 and M_2 is denoted by $M_1 \otimes M_2$. For a linear map $A: \mathcal{X} \rightarrow \mathcal{Y}$, $\ker A := \{x \in \mathcal{X} | Ax = 0\}$, and $\text{im } A := \{Ax | x \in \mathcal{X}\}$ will denote the kernel and image of A , respectively. For a real inner product space $\mathcal{X}$, if $\mathcal{V}$ is a subspace of $\mathcal{X}$, then $\mathcal{V}^\perp$ will denote the orthogonal complement of $\mathcal{V}$.

In this note, a weighted directed graph is denoted by $\mathcal{G} = (\mathcal{N}, \mathcal{E}, \mathcal{A})$, where $\mathcal{N} = \{1, 2, \dots, N\}$ is a finite nonempty set of nodes, $\mathcal{E} \subset \mathcal{N} \times \mathcal{N}$ is an edge set of ordered pairs of nodes, and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N}$ denotes the adjacency matrix. The (j, i) th entry a_{ji} is the weight associated with the edge (i, j) . We have $a_{ji} \neq 0$ if and only if $(i, j) \in \mathcal{E}$. Otherwise $a_{ji} = 0$. An edge $(i, j) \in \mathcal{E}$ designates that the information flows from node i to node j . A graph is said to be undirected if it has the property that $(i, j) \in \mathcal{E}$ implies $(j, i) \in \mathcal{E}$ for all $i, j \in \mathcal{N}$. We will assume that the graph is simple, i.e., $a_{ii} = 0$ for all $i \in \mathcal{N}$. For an edge (i, j) , node i is called the parent node, node j the child node and j is a neighbor of i . A directed path from node i_1 to i_l is a sequence of edges (i_k, i_{k+1}) , $k = 1, 2, \dots, l-1$ in the graph. A directed graph $\mathcal{G}$ is strongly connected if between any pair of distinct nodes i and j in $\mathcal{G}$, there exists a directed path from i to j , $i, j \in \mathcal{N}$.

The Laplacian $\mathcal{L} = [l_{ij}] \in \mathbb{R}^{N \times N}$ of $\mathcal{G}$ is defined as $\mathcal{L} := \mathcal{D} - \mathcal{A}$, where the i th diagonal entry of the diagonal matrix $\mathcal{D}$ is given by $d_i = \sum_{j=1}^N a_{ij}$. By construction, $\mathcal{L}$ has a zero eigenvalue with a corresponding eigenvector $\mathbf{1}_N$ (i.e., $\mathcal{L}\mathbf{1}_N = 0_N$), and if the graph is strongly connected, all the other eigenvalues lie in the open right-half complex plane.

For strongly connected graphs $\mathcal{G}$, we review the following lemma.

Lemma 1 ([19]–[21]): Assume $\mathcal{G}$ is a strongly connected directed graph. Then, there exists a unique positive row vector $r = [r_1, \dots, r_N]$ such that $r\mathcal{L} = 0$ and $r\mathbf{1}_N = N$. Define $R := \text{diag}\{r_1, \dots, r_N\}$. Then, $\hat{\mathcal{L}} := R\mathcal{L} + \mathcal{L}^T R$ is positive semidefinite, $\mathbf{1}_N^T \hat{\mathcal{L}} = 0$ and $\hat{\mathcal{L}}\mathbf{1}_N = 0$.

We note that $R\mathcal{L}$ is the Laplacian of the balanced digraph obtained by adjusting the weights in the original graph. The matrix $\hat{\mathcal{L}}$ is the Laplacian of the undirected graph obtained by taking the union of the

edges and their reversed edges in this balanced digraph. This undirected graph is called the mirror of this balanced graph [19].

B. Problem Formulation

In this note, we consider the following continuous-time LTI system

$$\begin{aligned} \dot{x} &= Ax \\ y &= Hx \end{aligned} \quad (1)$$

where $x \in \mathbb{R}^n$ is the state and $y \in \mathbb{R}^p$ is the measurement output. We partition the output y as $y = \text{col}(y_1, \dots, y_N)$, where $y_i \in \mathbb{R}^{p_i}$ and $\sum_{i=1}^N p_i = p$. Accordingly, $H = \text{col}(H_1, \dots, H_N)$ with $H_i \in \mathbb{R}^{p_i \times n}$. Here, the portion $y_i = H_i x \in \mathbb{R}^{p_i}$ is assumed to be the only information that can be acquired by the node i .

In this note, a standing assumption will be that the communication graph is a strongly connected directed graph. We will also assume that the pair (H, A) is observable. However, (H_i, A) is not assumed to be observable or detectable.

We will design a distributed Luenberger observer for the system given by (1) with the given communication network. The distributed observer will consist of N local observers, and the local observer at node i has the following dynamics:

$$\dot{\hat{x}}_i = A\hat{x}_i + L_i(y_i - H_i\hat{x}_i) + \gamma M_i \sum_{j=1}^N a_{ij}(\hat{x}_j - \hat{x}_i), \quad i \in \mathcal{N} \quad (2)$$

where $\hat{x}_i \in \mathbb{R}^n$ is the state of the local observer at node i , a_{ij} is the (i, j) th entry of the adjacency matrix $\mathcal{A}$ of the given network, $\gamma \in \mathbb{R}$ is a coupling gain to be designed, and $L_i \in \mathbb{R}^{n \times p_i}$ and $M_i \in \mathbb{R}^{n \times n}$ are gain matrices to be designed.

The objective of distributed state estimation is to design a network of observers that cooperatively estimate the state of the system described by system (1). We shall borrow the following definition from [14] for our analysis.

Definition 2: A distributed observer achieves *omniscience asymptotically* if for all initial conditions on (1) and (2) we have

$$\lim_{t \rightarrow \infty} \|\hat{x}_i(t) - x(t)\| = 0 \quad (3)$$

for all $i \in \mathcal{N}$, i.e., the state estimate maintained by each node asymptotically converges to the true state of the plant.

To analyze and synthesize observer (2), we define the local estimation error of the i th observer as

$$e_i := \hat{x}_i - x. \quad (4)$$

Combining (1) and (2) yields the following error equation:

$$\dot{e}_i = (A - L_i H_i)e_i + \gamma M_i \sum_{j=1}^N a_{ij}(e_j - e_i). \quad (5)$$

Let $e := \text{col}(e_1^T, e_2^T, \dots, e_N^T)$ be the joint vector of errors. Then, we have the global error system

$$\dot{e} = \Lambda e - \gamma \bar{M}(\mathcal{L} \otimes I_n)e \quad (6)$$

where

$$\Lambda = \text{diag}\{A - L_1 H_1, \dots, A - L_N H_N\}$$

$$\bar{M} = \text{diag}\{M_1, \dots, M_N\}.$$

The distributed observer achieves omniscience asymptotically (3) if and only if global error system (6) is stable.

Since (H_i, A) is not necessarily observable or detectable, L_i cannot be designed using any classical method directly. We use an orthogonal

transformation that yields the observability decomposition for the pair (H_i, A) . For $i \in \mathcal{N}$, let T_i be an orthogonal matrix, i.e., a square matrix such that $T_i T_i^T = I_n$, such that the matrices A and H_i are transformed by the state-space transformation T_i into the form

$$T_i^T A T_i = \begin{bmatrix} A_{io} & 0 \\ A_{ir} & A_{iu} \end{bmatrix}, H_i T_i = [H_{io} \quad 0] \quad (7)$$

where $H_{io} \in \mathbb{R}^{p_i \times v_i}$, $A_{io} \in \mathbb{R}^{v_i \times v_i}$, $A_{ir} \in \mathbb{R}^{(n-v_i) \times v_i}$, $A_{iu} \in \mathbb{R}^{(n-v_i) \times (n-v_i)}$, and $n - v_i$ is the dimension of the unobservable subspace of the pair (H_i, A) . Clearly, by construction, the pair (H_{io}, A_{io}) is observable. In addition, if we partition $T_i = [T_{i1} \ T_{i2}]$, where T_{i1} consists of the first v_i columns of T_i , then the unobservable subspace is given by $\text{im } T_{i2} = \ker O_i$, where $O_i = \text{col}(H_i, H_i A, \dots, H_i A^{n-1})$. Note that $\text{im } T_{i1} = \ker O_i^\perp$.

III. MAIN RESULTS

In this section, we study the distributed observer design. Before presenting the main design procedure, we state the following lemmas, based on Lemma 1. Our first lemma is standard.

Lemma 3 ([22]): For a strongly connected directed graph $\mathcal{G}$, zero is a simple eigenvalue of $\hat{\mathcal{L}} = R\mathcal{L} + \mathcal{L}^T R$ introduced in Lemma 1. Furthermore, its eigenvalues can be ordered as $\lambda_1 = 0 < \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_N$. Furthermore, there exists an orthogonal matrix $U = [\frac{1}{\sqrt{N}} \mathbf{1}_N \ U_2]$, where $U_2 \in \mathbb{R}^{N \times (N-1)}$, such that $U^T (R\mathcal{L} + \mathcal{L}^T R) U = \text{diag}\{0, \lambda_2, \dots, \lambda_N\}$.

Lemma 4: Let $\mathcal{L}$ be the Laplacian matrix associated with the strongly connected directed graph $\mathcal{G}$. For all $g_i > 0$, $i \in \mathcal{N}$, there exists $\epsilon > 0$ such that

$$T^T (\hat{\mathcal{L}} \otimes I_n) T + G > \epsilon I_{nN} \quad (8)$$

where $T = \text{diag}\{T_1, \dots, T_N\}$, $\hat{\mathcal{L}}$ is defined as in Lemma 1, $G = \text{diag}\{G_1, \dots, G_N\}$, and

$$G_i = \begin{bmatrix} g_i I_{v_i} & 0 \\ 0 & 0_{n-v_i} \end{bmatrix}, \quad i \in \mathcal{N}.$$

Proof: Inequality (8) holds if and only if the following inequality holds:

$$(U^T \hat{\mathcal{L}} U) \otimes I_n + (U^T \otimes I_n) T G T^T (U \otimes I_n) > 0 \quad (9)$$

where U is as in Lemma 3.

Inequality (9) holds if the following inequality holds:

$$\lambda_2 I_{nN} - (U \otimes I_n) \begin{bmatrix} \lambda_2 I_n & 0 \\ 0 & 0_{n(N-1)} \end{bmatrix} (U^T \otimes I_n) + T G T^T > 0. \quad (10)$$

Since $U = [\frac{1}{\sqrt{N}} \mathbf{1}_N \ U_2]$ and $U^T = [\frac{1}{\sqrt{N}} \mathbf{1}_N^T \ U_2^T]$, inequality (10) is equivalent to

$$\lambda_2 I_{nN} - \frac{\lambda_2}{N} (\mathbf{1}_N \otimes I_n) (\mathbf{1}_N^T \otimes I_n) + T G T^T > 0. \quad (11)$$

By pre- and postmultiplying with T^T and T , inequality (11) is equivalent to

$$\lambda_2 I_{nN} - \frac{\lambda_2}{N} T^T (\mathbf{1}_N \otimes I_n) (\mathbf{1}_N^T \otimes I_n) T + G > 0 \quad (12)$$

that is

$$\lambda_2 I_{nN} + G - \frac{\lambda_2}{N} [T_1 \ \dots \ T_N]^T [T_1 \ \dots \ T_N] > 0. \quad (13)$$

The orthogonal matrix T_i can be partitioned as $T_i = [T_{i1} \ T_{i2}]$, $i \in \mathcal{N}$. It is clear that $T_{i1} T_{i1}^T + T_{i2} T_{i2}^T = I_n$. By using the Schur complement lemma [23], inequality (13) is equivalent to

$$\begin{bmatrix} \Psi_1 & \dots & 0 & T_1^T \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \Psi_N & T_N^T \\ T_1 & \dots & T_N & \frac{N}{\lambda_2} I_n \end{bmatrix} > 0 \quad (14)$$

where $\Psi_i = \begin{bmatrix} (\lambda_2 + g_i) I_{v_i} & 0 \\ 0 & \lambda_2 I_{n-v_i} \end{bmatrix}$, $i \in \mathcal{N}$. Again using the Schur complement lemma, this is equivalent with

$$\begin{bmatrix} \lambda_2 I_{n-v_1} & 0 & \dots & 0 & T_{12}^T \\ 0 & \Psi_2 & \dots & 0 & T_2^T \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \Psi_N & T_N^T \\ T_{12} & T_2 & \dots & T_N & \frac{N}{\lambda_2} I_n - \frac{1}{\lambda_2 + g_1} T_{11} T_{11}^T \end{bmatrix} > 0. \quad (15)$$

By repeatedly using the Schur complement lemma, we finally obtain that inequality (13) holds if and only if

$$\frac{N}{\lambda_2} I_n - \sum_{i=1}^N \frac{1}{\lambda_2 + g_i} T_{i1} T_{i1}^T - \sum_{i=1}^N \frac{1}{\lambda_2} T_{i2} T_{i2}^T > 0. \quad (16)$$

The left-hand side of inequality (16) is equal to

$$\begin{aligned} & \frac{N}{\lambda_2} I_n - \sum_{i=1}^N \frac{1}{\lambda_2 + g_i} T_{i1} T_{i1}^T - \sum_{i=1}^N \frac{1}{\lambda_2} T_{i2} T_{i2}^T \\ &= \frac{N}{\lambda_2} I_n - \sum_{i=1}^N \frac{1}{\lambda_2} T_{i2} T_{i2}^T - \sum_{i=1}^N \frac{1}{\lambda_2} T_{i1} T_{i1}^T \\ & \quad + \sum_{i=1}^N \frac{1}{\lambda_2} T_{i1} T_{i1}^T - \sum_{i=1}^N \frac{1}{\lambda_2 + g_i} T_{i1} T_{i1}^T \\ &= \sum_{i=1}^N \left(\frac{1}{\lambda_2} - \frac{1}{\lambda_2 + g_i} \right) T_{i1} T_{i1}^T \\ &\geq \sum_{i=1}^N \left(\frac{1}{\lambda_2} - \frac{1}{\lambda_2 + g_{\min}} \right) T_{i1} T_{i1}^T \\ &= \left(\frac{N}{\lambda_2} - \frac{N}{\lambda_2 + g_{\min}} \right) [T_{11} \ \dots \ T_{N1}] [T_{11} \ \dots \ T_{N1}]^T \end{aligned} \quad (17)$$

where $g_{\min}$ is the minimum value of g_i , $i \in \mathcal{N}$. Obviously, we have $(\frac{N}{\lambda_2} - \frac{N}{\lambda_2 + g_{\min}}) > 0$ since $g_{\min} > 0$.

We will now prove that $\text{rank}[T_{11} \ T_{21} \ \dots \ T_{N1}] = n$, so that it has full row rank.

Indeed, for T_{i1} , we have

$$\text{im } T_{i1} = \ker O_i^\perp \quad (18)$$

where $O_i = \text{col}(H_i, H_i A, \dots, H_i A^{n-1})$.

Hence

$$\begin{aligned}
 \text{im}[T_{11} \quad T_{21} \quad \cdots \quad T_{N1}]^\perp &= (\text{im } T_{11} + \text{im } T_{21} + \cdots + \text{im } T_{N1})^\perp \\
 &= \bigcap_{i=1}^N \text{im } T_{i1}^\perp \\
 &= \bigcap_{i=1}^N \ker O_i \\
 &= \ker \begin{bmatrix} O_1 \\ \vdots \\ O_N \end{bmatrix} \\
 &= 0
 \end{aligned} \tag{19}$$

where we have used our standing assumption that the pair (H, A) is observable. This implies

$$\text{rank}[T_{11} \quad T_{21} \quad \cdots \quad T_{N1}] = n. \tag{20}$$

Consequently, $[T_{11} \quad T_{21} \quad \cdots \quad T_{N1}]$ has full row rank n , so we obtain

$$\left(\frac{N}{\lambda_2} - \frac{N}{\lambda_2 + g_{\min}} \right) [T_{11} \quad \cdots \quad T_{N1}] [T_{11} \quad \cdots \quad T_{N1}]^T > 0. \tag{21}$$

We conclude that the left-hand side of (8) is positive definite, and consequently, for any choice of $g_i > 0$, $i \in \mathcal{N}$, there exists a scalar $\epsilon > 0$ such that inequality (8) holds. ■

The following lemma now deals with the existence of a distributed observer of the form (2) that achieves omniscience with an *a priori* given error decay rate. A condition for its existence is expressed in terms of solvability of an LMI. Solutions to the LMI yield required gain matrices. Let $r_i > 0$, $i \in \mathcal{N}$, be as in Lemma 1. Let $g_i > 0$, $i \in \mathcal{N}$, and $\epsilon > 0$ be such that (8) holds. Let $\gamma \in \mathbb{R}$. Finally, let $\alpha > 0$ be the desired error decay rate. We have the following.

Lemma 5: There exist gain matrices L_i and M_i , $i \in \mathcal{N}$, such that the distributed observer (2) achieves omniscience asymptotically (3) and all solutions of error system (6) converge to zero with decay rate at least α if there exist positive definite matrices $P_{io} \in \mathbb{R}^{v_i \times v_i}$, $P_{iu} \in \mathbb{R}^{(n-v_i) \times (n-v_i)}$, and a matrix $W_i \in \mathbb{R}^{v_i \times p_i}$ such that

$$\begin{bmatrix} \Phi_i + \frac{\gamma}{r_i} g_i I_{v_i} & A_{ir}^T P_{iu} \\ P_{iu} A_{ir} & \text{Sym}(P_{iu} A_{iu}) + 2\alpha P_{iu} \end{bmatrix} - \frac{\gamma}{r_i} \epsilon I_n < 0 \quad \forall i \in \mathcal{N} \tag{22}$$

where $\Phi_i := P_{io} A_{io} + A_{io}^T P_{io} - W_i H_{io} - H_{io}^T W_i^T + 2\alpha P_{io}$. In that case, the gain matrices in the distributed observer (2) can be taken as

$$L_i := T_i \begin{bmatrix} L_{io} \\ 0 \end{bmatrix}, \quad M_i := T_i \begin{bmatrix} P_{io}^{-1} & 0 \\ 0 & P_{iu}^{-1} \end{bmatrix} T_i^T \tag{23}$$

where $L_{io} = P_{io}^{-1} W_i$, $i \in \mathcal{N}$.

Proof: Choose a candidate Lyapunov function for error system (6)

$$V(e_1, \dots, e_N) := \sum_{i=1}^N r_i e_i^T P_i e_i \tag{24}$$

where $P_i := T_i \begin{bmatrix} P_{io} & 0 \\ 0 & P_{iu} \end{bmatrix} T_i^T$. Clearly then $P_i > 0$.

The time-derivative of V is

$$\begin{aligned}
 \dot{V}(e) &= e^T ((R \otimes I_n) (P\Lambda + \Lambda^T P)) e \\
 &\quad - \gamma e^T (P\bar{M}(R\mathcal{L} \otimes I_n) + (\mathcal{L}^T R \otimes I_n) \bar{M}^T P) e \tag{25}
 \end{aligned}$$

where $P = \text{diag}\{P_1, \dots, P_N\}$. Since the matrix M_i in (23) is chosen as $M_i = P_i^{-1}$, we have $\bar{M} = P^{-1}$. Hence, the time-derivative of V becomes

$$\dot{V}(e) = e^T ((R \otimes I_n) (P\Lambda + \Lambda^T P) - \gamma \hat{\mathcal{L}} \otimes I_n) e \tag{26}$$

where, as before, $\hat{\mathcal{L}} = R\mathcal{L} + \mathcal{L}^T R$.

On the other hand, we get the following inequality by (22) and (8) in Lemma 4.

$$\text{diag}\{Q_1, \dots, Q_N\} - T^T \gamma (\hat{\mathcal{L}} \otimes I_n) T < 0 \tag{27}$$

where $Q_i = r_i \begin{bmatrix} \Phi_i & A_{ir}^T P_{iu} \\ P_{iu} A_{ir} & \text{Sym}(P_{iu} A_{iu}) + 2\alpha P_{iu} \end{bmatrix}$, $i \in \mathcal{N}$, with Φ_i as defined in the statement of the lemma.

Since $L_{io} = P_{io}^{-1} W_i$, pre- and postmultiplying inequality (27) with T and its transpose, we get

$$(R \otimes I_n) (P\Lambda + \Lambda^T P + 2\alpha P) - \gamma \hat{\mathcal{L}} \otimes I_n < 0 \tag{28}$$

which implies $\dot{V}(e) < -2\alpha V(e)$. Hence, the solutions of error system (6) converge to zero asymptotically with decay rate at least α [24], and the distributed observer (2) achieves omniscience asymptotically. ■

Using the previous lemmas we now arrive at our main result.

Theorem 6: Let $\alpha > 0$. If (H, A) is observable and $\mathcal{G}$ is a strongly connected directed graph, then there exists a distributed observer (2) that achieves omniscience asymptotically while all solutions of the error system converge to zero with decay rate at least α . Such observer is obtained as follows.

1) For each $i \in \mathcal{N}$, choose an orthogonal matrix T_i such that

$$T_i^T A T_i = \begin{bmatrix} A_{io} & 0 \\ A_{ir} & A_{iu} \end{bmatrix}, \quad H_i T_i = [H_{io} \quad 0] \tag{29}$$

with (H_{io}, A_{io}) observable.

2) Compute the positive row vector $r = [r_1, \dots, r_N]$ such that $r\mathcal{L} = 0$ and $r\mathbf{1}_N = N$.

3) Put $g_i = 1$, $i \in \mathcal{N}$ and take $\epsilon > 0$ such that (8) holds.

4) Take $\gamma > 0$ sufficiently large so that for all $i \in \mathcal{N}$

$$A_{iu} + A_{iu}^T - \left(\frac{\gamma}{r_i} \epsilon - 2\alpha \right) I_{n-v_i} + \frac{1}{\frac{\gamma}{r_i} \epsilon - 2\alpha} A_{ir} A_{ir}^T < 0. \tag{30}$$

5) Choose L_{io} such that all eigenvalues of $A_{io} - L_{io} H_{io}$ lie in the region $\{s \in \mathbb{C} \mid \text{Re}(s) < -\alpha\}$.

6) For all $i \in \mathcal{N}$, solve the Lyapunov equation

$$\text{Sym}(P_{io}(A_{io} - L_{io} H_{io} + \alpha I_{v_i})) + \left(\frac{\gamma}{r_i} - 2\alpha \right) I_{v_i} = 0 \tag{31}$$

to obtain $P_{io} > 0$.

7) Define

$$L_i := T_i \begin{bmatrix} L_{io} \\ 0 \end{bmatrix}, \quad M_i := T_i \begin{bmatrix} P_{io}^{-1} & 0 \\ 0 & I_{n-v_i} \end{bmatrix} T_i^T, \quad i \in \mathcal{N}. \tag{32}$$

Proof: We choose $g_i = 1$, $i \in \mathcal{N}$. Since the pair (H, A) is observable and the graph $\mathcal{G}$ is a strongly connected directed graph, $\epsilon > 0$ can be obtained by Lemma 4.

Putting $P_{iu} = I_{n-v_i}$, $i \in \mathcal{N}$, inequality (22) in Lemma 5 becomes

$$\begin{bmatrix} \Phi_i + \frac{\gamma}{r_i} I_{v_i} & A_{ir}^T \\ A_{ir} & A_{iu} + A_{iu}^T + 2\alpha I_{n-v_i} \end{bmatrix} - \frac{\gamma}{r_i} \epsilon I_n < 0 \quad \forall i \in \mathcal{N}. \quad (33)$$

where $\Phi_i = P_{io} A_{io} + A_{io}^T P_{io} - W_i H_{io} - H_{io}^T W_i^T + 2\alpha P_{io}$. By substituting (31) and $W_i = P_{io} L_{io}$ into (33), we have that inequality (33) holds if

$$\begin{bmatrix} -(\frac{\gamma}{r_i} \epsilon - 2\alpha) I_{v_i} & A_{ir}^T \\ A_{ir} & \Psi_i \end{bmatrix} < 0 \quad \forall i \in \mathcal{N} \quad (34)$$

where $\Psi_i = \text{Sym}(A_{iu}) - (\frac{\gamma}{r_i} \epsilon - 2\alpha) I_{n-v_i}$.

By using the Schur complement lemma, (34) is equivalent with

$$A_{iu} + A_{iu}^T - \left(\frac{\gamma}{r_i} \epsilon - 2\alpha \right) I_{n-v_i} + \frac{1}{\frac{\gamma}{r_i} \epsilon - 2\alpha} A_{ir} A_{ir}^T < 0 \quad \forall i \in \mathcal{N}. \quad (35)$$

As stated in step 4, inequality (35) holds with sufficiently large $\gamma > 0$.

Thus, we find that the parameters introduced in steps 3–6 guarantee that inequality (22) in Lemma 5 holds. Hence, the distributed observer (2) with gain matrices L_i and M_i and coupling gain γ achieves omniscience asymptotically with decay rate at least α . ■

Remark 7: Points 3 and 4 in Theorem 6 should read as follows. Choose g_i s in (8) all equal to 1. According to Lemma 4, there exists $\epsilon > 0$ (sufficiently small) such that (8) holds. Compute this ϵ , plug it into (30), and solve this inequality to find a γ . This amounts to taking γ sufficiently large. One could argue that one does not really need to compute ϵ , since inequality (30) will hold anyway for γ sufficiently large. Note, however, that one does not know *how large* γ should actually be. Point 4 tells exactly how large γ should be taken, and for this, one first needs to compute ϵ in (8).

Remark 8: Since (H_{io}, A_{io}) is observable, the Lyapunov equation (31) in step 6 can indeed be made to have a solution for any $\alpha > 0$. The coupling gain $\gamma > 0$ can be taken sufficiently large such that (30) holds for any given $\alpha > 0$, which means that the error can be made to converge to zero at any desired rate.

Remark 9: The design procedure in Theorem 6 gives one possible choice of solutions of inequality (22) in Lemma 5, which also means that inequality (22) always has the required solutions under our standing assumptions that (H, A) is observable and the graph $\mathcal{G}$ is strongly connected. In fact, inequalities (8) in Lemma 4 and (22) in Lemma 5 both are LMIs, which can be solved numerically by using the LMI Toolbox or YALMIP in MATLAB directly.

Remark 10: In the special case that the communication graph among the observers is a connected undirected graph, we have that $r = \mathbf{1}_N^T$ is the unique positive row vector such that $r\mathcal{L} = 0$ and $r\mathbf{1}_N = N$. In the design procedure of Theorem 6, we can then take $r_i = 1$ for all $i \in \mathcal{N}$.

Remark 11: Depending on the application at hand, exact knowledge of the Laplacian matrix $\mathcal{L}$ or the left eigenvector r might not be available. In that case, however, the algorithm of Theorem 6 is still applicable. The idea in this case is to choose the coupling gain γ sufficiently large. More concrete, suppose that the Laplacian $\mathcal{L}$ and the exact values of r_i are unknown but that we know upper and lower bounds, say $r_i \in [r_{\min}, r_{\max}]$, $\forall i \in \mathcal{N}$. Then, first choose $\epsilon > 0$ sufficiently small as in step 3. Next, use $r_{\max}$ and $r_{\min}$ instead of r_i in steps 4 and 6, respectively, to design a suitable distributed observer. Below we provide a short analysis of this procedure. Clearly, (30) holds if the

Fig. 2. Communication graph among nodes.

following inequality holds:

$$A_{iu} + A_{iu}^T - \left(\frac{\gamma}{r_{\max}} \epsilon - 2\alpha \right) I_{n-v_i} + \frac{1}{\frac{\gamma}{r_{\max}} \epsilon - 2\alpha} A_{ir} A_{ir}^T < 0. \quad (36)$$

Choose γ sufficiently large so that (36) holds. Next, solve

$$\text{Sym}(P_{io}(A_{io} - L_{io} H_{io} + \alpha I_{v_i})) + \left(\frac{\gamma}{r_{\min}} - 2\alpha \right) I_{v_i} = 0 \quad (37)$$

to obtain P_{io} . Clearly then, also

$$\text{Sym}(P_{io}(A_{io} - L_{io} H_{io} + \alpha I_{v_i})) + \left(\frac{\gamma}{r_i} - 2\alpha \right) I_{v_i} \leq 0. \quad (38)$$

In the proof of Theorem 6, it is then easily verified that (33) holds if (36) and (37) hold. Hence, the LMI (22) has the required solutions and the proposed distributed observer indeed works.

IV. SIMULATION EXAMPLE

In this section, we will use a numerical example borrowed from [25] to illustrate the effectiveness of our approach.

Consider the observed system

$$\dot{x} = Ax, y = Hx \quad (39)$$

where

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 \\ 1 & -2 & -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ -8 & 1 & -1 & -1 & -2 & 0 \\ 4 & -0.5 & 0.5 & 0 & 0 & -4 \end{bmatrix}$$

$$H = \begin{bmatrix} 1 & 0 & 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & 4 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} H_1 \\ H_2 \\ H_3 \\ H_4 \end{bmatrix}.$$

The communication network is given by the strongly connected digraph in Fig. 2. The Laplacian of this graph is given by

$$\mathcal{L} = \begin{bmatrix} 2 & -1 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ -1 & -1 & 2 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}.$$

It can be seen that none of the local systems (H_i, A) is observable, but (H, A) is an observable pair. We will apply the conceptual

algorithm of Theorem 6 to design a distributed observer. The normalized positive left eigenvector of the Laplacian is computed to be $r = [0.8 \ 1.6 \ 0.8 \ 0.8]$. We choose a decay rate $\alpha = 1$. A coupling gain is computed to be $\gamma = 219.7065$. The local observer gain matrices are computed as

$$L_1 = \begin{bmatrix} 0.1661 & 0.1661 \\ 0 & 0 \\ 0 & 0 \\ 0.4982 & -0.4982 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, L_2 = \begin{bmatrix} 3.5770 \\ -14.3468 \\ 0.5188 \\ -5.4729 \\ -14.8217 \\ -5.9758 \end{bmatrix}$$

$$L_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, L_4 = \begin{bmatrix} -2.2551 & 0.3970 \\ -8.9847 & -2.6908 \\ 1.6216 & 0.1859 \\ 6.2723 & 0.9189 \\ -3.0056 & -1.7857 \\ 15.9378 & -1.8280 \end{bmatrix}$$

$$M_1 = \begin{bmatrix} 0.0037 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0037 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} 0.0512 & 0.0879 & -0.1315 & -0.1416 & -0.0429 & 0.0297 \\ 0.0879 & 0.1796 & -0.2330 & -0.2519 & -0.0576 & 0.0528 \\ -0.1315 & -0.2330 & 0.3450 & 0.3644 & 0.1031 & -0.0822 \\ -0.1416 & -0.2519 & 0.3644 & 0.3981 & 0.1173 & -0.0816 \\ -0.0429 & -0.0576 & 0.1031 & 0.1173 & 0.0549 & -0.0216 \\ 0.0297 & 0.0528 & -0.0822 & -0.0816 & -0.0216 & 0.0259 \end{bmatrix}$$

$$M_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0130 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M_4 = \begin{bmatrix} 0.0390 & 0.0293 & -0.0195 & -0.1186 & -0.1410 & 0.0409 \\ 0.0293 & 0.0315 & -0.0161 & -0.0932 & -0.1002 & 0.0303 \\ -0.0195 & -0.0161 & 0.0103 & 0.0598 & 0.0696 & -0.0210 \\ -0.1186 & -0.0932 & 0.0598 & 0.3643 & 0.4299 & -0.1273 \\ -0.1410 & -0.1002 & 0.0696 & 0.4299 & 0.5260 & -0.1603 \\ 0.0409 & 0.0303 & -0.0210 & -0.1273 & -0.1603 & 0.0621 \end{bmatrix}$$

Fig. 3. State components of the observed plant and their estimates.

Fig. 4. Estimation errors associated with state component 1.

Fig. 5. Estimation errors associated with state component 2.

In the simulation, the initial state of the observed system is taken as $x(0) = [1 \ 3 \ -2 \ -3 \ -1 \ 2]^T$. For each local observer the initial state is taken to be zero.

The state components and their estimates are depicted in Fig. 3. It can be seen that all estimates converge to the actual state

Fig. 6. Estimation errors associated with state component 3.

Fig. 9. Estimation errors associated with state component 6.

Fig. 7. Estimation errors associated with state component 4.

Fig. 8. Estimation errors associated with state component 5.

components. Figs. 4–9 show the estimation errors associated with each state component. The errors converge to zero at the desired rate.

V. CONCLUSION

In this note, we have proposed a novel simple approach to distributed observer design for LTI systems. The information among the local observers is exchanged by a strongly connected directed graph. The local Luenberger observer at each node is designed to asymptotically estimate the state of the dynamical system. Each local observer has state dimension equal to that of the observed plant. We have analyzed the structure of the required gain matrices using an observability decomposition of the local systems. By introducing an auxiliary undirected graph, a bank of LMIs is presented to calculate the gain matrices in our distributed observer. Finally, we have recovered the known sufficient conditions for the existence of a distributed observer and have presented a simple optional algorithm to design a distributed observer. In future research, we plan to focus on order reduction of the distributed observer, and deal with the problem of design of minimal order distributed observers. A alternative research direction is to allow the local observers to estimate (instead of the full state) only part of the state of the observed system.

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