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Published in:
Automatica

DOI:
[10.1016/j.automatica.2019.108505](https://doi.org/10.1016/j.automatica.2019.108505)

IMPORTANT NOTE: You are advised to consult the publisher's version (publisher's PDF) if you wish to cite from it. Please check the document version below.

Document Version
Publisher's PDF, also known as Version of record

Publication date:
2019

[Link to publication in University of Groningen/UMCG research database](#)

Citation for published version (APA):

Chen, Y., Xia, W., Cao, M., & Lu, J. (2019). Random asynchronous iterations in distributed coordination algorithms. *Automatica*, 109, [108505]. <https://doi.org/10.1016/j.automatica.2019.108505>

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Brief paper

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ARTICLE INFO

Article history:

Received 28 June 2017

Received in revised form 14 February 2019

Accepted 10 July 2019

Available online 13 August 2019

Keywords:

Distributed coordination algorithm

Asynchronous iteration

Random process

Consensus

ABSTRACT

Distributed coordination algorithms (DCA) carry out information processing processes among a group of networked agents without centralized information fusion. Though it is well known that DCA characterized by an SIA (stochastic, indecomposable, aperiodic) matrix generate consensus asymptotically via synchronous iterations, the dynamics of DCA with asynchronous iterations have not been studied extensively, especially when viewed as stochastic processes. This paper aims to show that for any given irreducible stochastic matrix, even non-SIA, the corresponding DCA lead to consensus successfully via random asynchronous iterations under a wide range of conditions on the transition probability. Particularly, the transition probability is neither required to be i.i.d, nor characterized by a Markov chain.

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1. Introduction

Distributed coordination algorithms (DCA) use local information of a group of networked agents to generate specific collective behaviours. Nowadays, DCA have been used not only to explain social or economic phenomena (DeGroot, 1974), but also to solve practical engineering problems (Nedić & Olshevsky, 2015). A typical DCA generates aligned collective motion, usually referred to as consensus (or synchronization, Meng, Yang, Li, Ren, and Wu (2018) and Wang, Ma, Zheng, and Chen (2017)).

When agents update their states using DCA, they may do so synchronously or asynchronously. More precisely, by synchronous updating we mean all the agents update their states at the same time repeatedly. This requires the availability of a global clock or a set of identical local clocks, which is a stringent requirement in practice (see, Dolev, Halpern, & Strong, 1986). By asynchronous updating we mean each agent has an independent local clock, according to which it updates its own state without paying attention to when the other agents update. Only in this way DCA can be constructed in a fully distributed manner (Cao,

Morse, & Anderson, 2008a). It has been recently reported that the synchronous and asynchronous implementation of DCA may lead to dramatically different asymptotic collective behaviours; for example, some DCA may converge under synchronous updating but diverge under asynchronous updating (Xia & Cao, 2014; Xia, Liu, Cao, Johansson, & Basar, 2019).

The possible significant differences of DCA in deterministic and stochastic settings are reflected in the fundamental differences in their corresponding analytical tools: the determination of convergence in DCA, even for a pair of stochastic matrices, has been proved to be NP-hard in a deterministic setting (Blondel & Olshevsky, 2014); in sharp comparison, the convergence for DCA with finite Markovian random switching modes can be determined by using the classic LMI-based techniques (Boukas & Yang, 1995), which have been proved to obtain desired solutions with high efficiency. Therefore, the analysis in stochastic settings often generates less conservative sufficient conditions for consensus: In 2007, Porfiri and Stilwell gave some sufficient conditions for consensus over random weighted directed graphs generated by i.i.d random variables (Porfiri & Stilwell, 2007). In 2013, You et al. established some necessary and sufficient conditions for consensus based on the assumption of Markovian switching topologies (You, Li, & Xie, 2013); Matei et al. gave some sufficient conditions for linear consensus problem under Markovian random graphs (Matei, Baras, & Somarakis, 2013). Note that the network topologies in these works are either Markovian or

[☆] The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Bert Tanner under the direction of Editor Christos G. Cassandras.

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generated by i.i.d random variables. When asynchronous updating in DCA is not generated by a Markovian chain or independent random variables, the analysis becomes much more challenging due to the limitations of the existing analytical methods.

In the deterministic setting, the topological condition for consensus of DCA with asynchronous iteration is generally very restrictive. In 2014, Xia and Cao proved that for any given scrambling matrix, the corresponding DCA reach consensus for any asynchronous iteration (Xia & Cao, 2014). It should be noted that any pair of nodes in the graph of a scrambling matrix share a common neighbour, which makes such a graph densely connected. In this paper, we will investigate the asynchronous iterations of DCA in the stochastic setting with the aim of relaxing the restrictive constraint on topological structures. To realize this purpose, we will transform the consensus problem of random asynchronous DCA to the random walk problem along a labelled directed cycle, and propose a graphical method to analyse the convergence of random asynchronous DCA. Specifically, the contributions of this paper can be summarized as follows:

- The convergence of asynchronous iterations of DCA is investigated in the stochastic setting for the first time. The obtained results only require the graph of the given matrix of DCA to be connected. Compared with the related results in the deterministic setting, we do not need to require the matrix to be SIA or scrambling;
- The critical conditions for consensus in the traditional stochastic setting, such as i.i.d or Markovian switching, existence of self-loops in the communication topology are not needed in our main result any more.

Moreover, removal of self-loops in the communication topology changes the computational complexity of the consensus problem and makes the analysis extremely difficult: as indicated in Blondel and Olshevsky (2014), the deciding of consensus is P when each node contains a self-loop; however, it is NP-hard when not every node contains a self-loop.

The rest of the paper is organized as follows: Section 2 gives some preliminaries on graph theory and formulates the problem of random asynchronous updating for DCA; Section 3 presents the main results and related discussions; Section 4 provides the skeleton of the technical proof (details are given in Chen, Xia, Cao, and Lü (2018a) due to page limit); Section 5 gives some numerical examples; and Section 6 concludes this paper.

2. Preliminaries and problem formulation

In this section, we will give some preliminaries on graph theory and asynchronous iterations of DCA in two subsections, respectively.

2.1. Preliminaries

A graph $\mathcal{G} = (V, \mathcal{E})$ is composed of two sets, where V is the set of nodes and $\mathcal{E} \subseteq V \times V$ is the set of edges. A path of $\mathcal{G}$ is composed of a sequence of distinct nodes $i_1, i_2, \dots, i_k$ which satisfy $(i_j, i_{j+1}) \in \mathcal{E}$ for any $1 \leq j \leq k-1$. $\mathcal{G}$ is called rooted if there exists a node $r \in V$ such that for any $j \neq r$, there is a path from r to j , where r is called a root of $\mathcal{G}$. $\mathcal{G}$ is called strongly connected if there exists a path from any node $i \in V$ to any node $j \in V$ ($i \neq j$). The collection of all the roots of graph $\mathcal{G}$ is defined as $\mathfrak{r}(\mathcal{G})$. If there exists an edge from node i to itself, we call such an edge a self-loop. Specifically, if $\mathfrak{r}(\mathcal{G}) = \{r\}$, we say node r is the unique root of graph $\mathcal{G}$.

In order to regroup the nodes in a graph, we assign a positive integer to each node of a graph, and these integers are called labels of the nodes. A graph $\mathcal{G} = (V, \mathcal{E})$ is said to be a labelled

Fig. 1. Transform a strongly connected graph to a labelled directed cycle.

graph if each node $i \in V$ is assigned with an integer label $\text{label}(i)$. It should be noted that there may exist two nodes $i, j \in V$ with identical labels, i.e., $\text{label}(i) = \text{label}(j)$. The following proposition tells us the fact: any strongly connected graph can be mapped to a labelled directed cycle.

Proposition 1. *Given any strongly connected graph $\mathcal{G} = (V, \mathcal{E})$ with N nodes, it can be mapped to a labelled directed cycle $\mathcal{C}$ with length $l \leq N(N-1)$ and contains all the nodes of V .*

Proof. Due to the strong connectivity of graph $\mathcal{G}$, there exists a path from k to $k+1$ ($1 \leq k \leq N-1$) with length no more than $N-1$, and there also exists a path from node N to 1 with length no more than $N-1$. Concatenate these N paths and one generates a cycle which contains all the nodes of V . $\square$

As shown in Fig. 1, a strongly connected graph with 4 nodes can be transformed to a labelled directed cycle with 6 nodes, in which $V = \{1, 2, 3, 4, 5, 6\}$ and the label of each node $i \in V$ is given in the bracket. In particular, nodes 2 and 5 share the same label 2, nodes 6 and 3 share the same label 4.

A matrix $A = (a_{ij})_{i,j=1}^N$ is called stochastic (or row stochastic) if $a_{ij} \geq 0$ ($\forall i, j \in V$) and $\sum_{j=1}^N a_{ij} = 1$ ($\forall i \in V$). The graph $\mathcal{G}(A) = (V, \mathcal{E})$ corresponding to A is defined by: $V = \{1, 2, \dots, N\}$, and $(j, i) \in \mathcal{E}$ if and only if $a_{ij} > 0$. Based on $\mathcal{G}(A)$ we define $\mathfrak{r}(A) = \mathfrak{r}(\mathcal{G}(A))$ and

$$\mathcal{N}(A, v) = \{k : a_{vk} > 0, k \in V\}$$

as the neighbour of node v in $\mathcal{G}(A)$. The ergodic coefficient of a stochastic matrix A is defined by

$$\lambda(A) = 1 - \min_{i \neq j} \sum_{k=1}^N \min(a_{ik}, a_{jk}).$$

A stochastic matrix A is called scrambling if $\lambda(A) < 1$, and is called SIA if $\lim_{k \rightarrow \infty} A^k = \mathbf{1}\xi^T$ for some $\xi \in \mathbb{R}^N$, where $\mathbf{1} \in \mathbb{R}^N$ is a vector with each entry being 1.

Given two stochastic matrices $A = (a_{ij})_{i,j=1}^N, B = (b_{ij})_{i,j=1}^N \in \mathbb{R}^{N \times N}$, we say A and B are of the same type if $\text{sgn}(a_{ij}) = \text{sgn}(b_{ij})$ holds for any $i, j = 1, 2, \dots, N$, denoted by $A \sim B$, where $\text{sgn}(\cdot)$ is the sign function.

Given a vector $x = (x_1, x_2, \dots, x_N)^T \in \mathbb{R}^N$, we define

$$\Delta(x) = \max_{i=1}^N x_i - \min_{i=1}^N x_i$$

as the maximal discrepancy of vector x . Given a stochastic matrix A and $y = Ax$ with $y \in \mathbb{R}^N$, it holds that $\Delta(y) \leq \lambda(A)\Delta(x)$. Cao, Morse, and Anderson (2008b)

Specifically, the ergodic coefficient $\lambda(\cdot)$ has the following important properties.

- Given any stochastic matrices $A_1, A_2 \in \mathbb{R}^{N \times N}$, it holds $\lambda(A_1 A_2) \leq \lambda(A_1) \lambda(A_2)$. Cao et al. (2008a)
- Given any stochastic matrix $A \in \mathbb{R}^{N \times N}$, it holds $\lambda(A) = \sup_{\Delta(x)=1} \Delta(Ax)$. Seneta (1981)

A matrix $A = (a_{ij})_{i,j=1}^N$ is called column stochastic if A^T is row stochastic. In this paper, we use a column stochastic matrix to denote a Markovian chain. The entry a_{ij} in a column stochastic matrix represents the transition probability from node j to node i . Without specific declaration, for any stochastic matrix in this paper, we mean it is row stochastic.

2.2. Problem formulation

DCA characterizes the evolution of states in a network of agents via local interaction. A typical one of DCA is the following linear averaging protocol

$$x_i(k+1) = \begin{cases} \sum_{j=1}^N a_{ij}x_j(k), & \text{if agent } i \text{ updates} \\ x_i(k), & \text{at time instant } k; \\ & \text{if agent } i \text{ does not} \\ & \text{update at time instant } k, \end{cases}$$

where $x_i(k)$ is the state of agent i at time k , a_{ij} denotes the coupling coefficient between agent i and j , $a_{ij} \geq 0$ for any $i, j \in V$, and $\sum_{j=1}^N a_{ij} = 1$ for any $i \in V$. Define $A = (a_{ij})_{i,j=1}^N \in \mathbb{R}^{N \times N}$ as the corresponding stochastic coupling matrix.

If each agent $i \in V$ updates at any time k , then the above DCA transforms to

$$x(k+1) = Ax(k), \quad k = 1, 2, \dots, \quad (1)$$

where

$$x(k) = (x_1(k), x_2(k), \dots, x_N(k))^T$$

represents the state vector for all the individuals. We call DCA (1) the synchronous DCA.

However, synchronous DCA is difficult to be implemented in practice since the local clocks associated with all the nodes are generally nonidentical (see Dolev et al. (1986), the impossibility of clock synchronization). In the case of asynchronous iteration, only part of the agents $\sigma_k \in 2^V$ (2^V is the set of all subsets of V) update their states at each time instant k , then one generates the following asynchronous DCA:

$$x(k+1) = A_{\sigma_k}x(k), \quad (2)$$

where A_{σ_k} is the asynchronous iteration matrix with the index set $\sigma_k \subseteq V$, whose definition is given by the following construction process: if $j \in \sigma_k$, then the j th row of A_{σ_k} equals the j th row of A ; if $j \notin \sigma_k$, then the j th row of A_{σ_k} is the j th elementary vector e_j . Specifically, when only one agent updates at time k , σ_k becomes a singleton and in this case we also use σ_k to denote the agent σ ($\sigma \in V$) that updates in the sequel. It follows that

$$A_{\sigma} \triangleq A_{\sigma_k} = (e_1, e_2, \dots, e_{\sigma-1}, a_{\sigma}^T, e_{\sigma+1}, \dots, e_N)^T,$$

where a_{σ} is the σ th row of A , e_j are elementary column vectors.

For example, given a stochastic matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0.2 & 0.8 & 0 \\ 0 & 0.7 & 0.3 \end{pmatrix},$$

then A_{σ} defines

$$A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0.2 & 0.8 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_{\{1,3\}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0.7 & 0.3 \end{pmatrix}.$$

Denote $\{A_{\sigma_k}\}_{k=1}^{\infty}$ as an asynchronous iteration sequence of matrix A . $\{A_{\sigma_k}\}_{k=1}^{\infty}$ is said to generate consensus in the deterministic setting if for any initial value $x(1) \in \mathbb{R}^N$, there exists $\xi \in \mathbb{R}$ such

that

$$\lim_{k \rightarrow \infty} x(k+1) = \xi$$

in DCA (2).

The convergence of an asynchronous iteration sequence of a stochastic matrix cannot be determined by the SIA property of this matrix: the asynchronous implementation of an SIA matrix may not generate consensus, but the asynchronous implementation of a non-SIA matrix may generate consensus (see, examples in Chen et al. (2018a) and Xia and Cao (2014)).

In 2014, Xia and Cao proposed the following sufficient condition for consensus of asynchronous DCA (2) in the deterministic setting.

Proposition 2. *If A is scrambling and there exists $q > 0$ such that $\bigcup_{k=j}^{j+q-1} \sigma_k = V$ for any $j \geq 1$, then the asynchronous iteration sequence $\{A_{\sigma_k}\}_{k=1}^{\infty}$ generates consensus.*

However, when we consider asynchronous iteration in the stochastic setting, i.e., $\{\sigma_k\}_{k \geq 1}$ are generated by random variables $\{\mathcal{E}_k\}_{k \geq 1}$, the dynamics of asynchronous DCA (2) will be quite different. A major challenge for the analysis is that the value of σ_k and the corresponding transition probability may depend on its historic values: $\sigma_{k-1}, \sigma_{k-2}, \dots, \sigma_1$. In this paper, we use $(\Omega, \mathcal{F}, \mathbb{P})$ to represent the probability space, where Ω is the sample space of σ_k , $\mathcal{F}$ is the σ -field, and $\mathbb{P}$ is the probability function.

When $\sigma_k \in 2^V$, the possible values of A_{σ_k} is $\mathfrak{A} = \{A_{\sigma} : \sigma \in 2^V\}$, where $|\mathfrak{A}| = 2^N$. When $\sigma_k \in V$, the possible values of A_{σ_k} is $\mathfrak{A} = \{A_{\sigma} : \sigma \in V\}$, where $|\mathfrak{A}| = N$. Since the diagonal entries of A may contain zeros, the set $\mathfrak{A}$ may also contain elements whose diagonal entries have some zeros. The existence of zero diagonal entries in the coupling matrices usually brings big challenges in analysing the products of stochastic matrices (see, Chen, Xiong, and Li (2016), Chen, Xia, Cao, and Lü (2018b), Touri and Nedić (2012), and Touri and Nedić (2014)).

In the stochastic setting, the asynchronous DCA (2) is said to realize consensus almost surely if

$$\lim_{k \rightarrow \infty} \mathbb{P} \left(\sum_{j=1}^N \left\| x_j(k) - \frac{1}{N} \sum_{i=1}^N x_i(k) \right\|^2 \geq \varepsilon \right) = 0 \quad (3)$$

for any $\varepsilon > 0$ and $x(1) \in \mathbb{R}^N$. If one defines the following projection matrix $P = I - \frac{1}{N} \mathbf{1}\mathbf{1}^T$, where I is the identity matrix, then (3) can be equivalently rewritten as

$$\lim_{k \rightarrow \infty} \mathbb{P} (\|Px(k)\| \geq \varepsilon) = 0 \quad (4)$$

for any $\varepsilon > 0$ and $x(1) \in \mathbb{R}^N$.

In the rest of this paper, for the simplicity of expression, we denote

$$k_1 : k_2 = \{k_1, k_1 - 1, \dots, k_2\},$$

$$\sigma_{k_1:k_2} = \{\sigma_{k_1}, \sigma_{k_1-1}, \dots, \sigma_{k_2}\},$$

$$A_{\sigma_{k_1:k_2}} = A_{\sigma_{k_1}} A_{\sigma_{k_1-1}} A_{\sigma_{k_1-2}} \cdots A_{\sigma_{k_2+1}} A_{\sigma_{k_2}}$$

for any $k_1 \geq k_2 \geq 1$. Particularly, it follows $\sigma_{k:k} = \{\sigma_k\}$ and in the case of $\sigma_k \in V$, we do not distinguish the expressions of σ_k and $\{\sigma_k\}$.

In the next section, we will give several sufficient conditions which guarantee almost surely consensus of asynchronous DCA (2).

3. Conditions for random asynchronous consensus

We briefly summarize the sufficient conditions for random asynchronous consensus as follows.

Theorem 1. The asynchronous DCA (2) generates consensus almost surely if all the following conditions hold:

- (a) $\mathcal{G}(A)$ is rooted and $\sigma_k \in 2^V$.
- (b) There exists $\alpha > 0$ such that if $\mathbb{P}(\sigma_k \mid \sigma_{(k-1):1}) \neq 0$, then $\mathbb{P}(\sigma_k \mid \sigma_{(k-1):1}) \geq \alpha$.
- (c) For any given past values of $\sigma_{(k-1):1}$, the set

$$\mathcal{S}_{\sigma_{(k-1):1}} = \{\sigma : \mathbb{P}(\sigma_k = \sigma \mid \sigma_{(k-1):1}) \neq 0\}$$

only depends on k but not the historic values $\sigma_{(k-1):1}$, i.e., there exists $\mathcal{S}_k$ such that

$$\mathcal{S}_{\sigma_{(k-1):1}} = \mathcal{S}_k, \quad \forall \sigma_{(k-1):1} \in \underbrace{2^V \times 2^V \cdots \times 2^V}_{k-1 \text{ times}}.$$

- (d) There exists $q > 0$ such that

$$\bigcup_{\tau=k}^{k+q-1} \left(\bigcup_{\sigma \in \mathcal{S}_\tau} \sigma \right) = V, \quad \forall k \geq 1.$$

- (e) There exists a strongly connected component χ of $\mathfrak{r}(A)$ such that for any $j \in \chi$, it holds $\mathcal{S}_k^j \neq \emptyset$ and

$$\chi \cap \left(\bigcap_{\sigma \in \mathcal{S}_k^j} \sigma \right) = j, \quad \forall k \geq 1,$$

where

$$\mathcal{S}_k^j = \{\sigma : \sigma \in \mathcal{S}_k \text{ and } j \in \sigma\}.$$

The meaning of the 5 conditions in Theorem 1 is intuitive: condition (a) gives the topological condition on $\mathcal{G}(A)$; condition (b) requires a positive infimum on all the nonzero transition probabilities; condition (c) is called historic independence of nonzero probabilities, which requires the set $\mathcal{S}_{\sigma_{(k-1):1}}$ to be independent of the historic values $\sigma_{(k-1):1}$; condition (d) is called the joint coverage condition, which means each node of V has a nonzero probability to be chosen to update in every consecutive q steps; condition (e) is called the quasi-singleton property of nodes in χ , which means once some node $j \in \chi$ has a nonzero probability to be chosen at time k , then the intersection of χ and all the possible values of σ_k which contain j is j itself.

To illustrate the meanings of the conditions (c)–(e) in Theorem 1, we give the following example:

Example 1. Suppose $\{\sigma_k\}_{k=1}^\infty$ are generated by i.i.d random variables, $V = \{1, 2, 3, 4\}$, and

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad (5)$$

where $\mathfrak{r}(A) = \{1, 2, 3\}$. Consider a subset of 2^V , such as¹

$$\mathcal{S}_k = \{\{1, 2, 4\}, \{1, 3, 4\}, \{2, 3\}\}, \quad \forall k \geq 1.$$

Since $\mathcal{S}_k$ does not rely on the historic values $\sigma_{(k-1):1}$, condition (c) naturally holds. Note that

$$\begin{aligned} \mathcal{S}_k^1 &= \{\{1, 2, 4\}, \{1, 3, 4\}\}, & \mathcal{S}_k^2 &= \{\{2, 3\}, \{1, 2, 4\}\}, \\ \mathcal{S}_k^3 &= \{\{1, 3, 4\}, \{2, 3\}\}, \end{aligned}$$

one can verify that

$$\bigcap_{\sigma \in \mathcal{S}_k^1} \sigma = \{1, 4\}, \quad \bigcap_{\sigma \in \mathcal{S}_k^2} \sigma = 2, \quad \bigcap_{\sigma \in \mathcal{S}_k^3} \sigma = 3.$$

¹ The set $\mathcal{S}_k$ can be set as time-varying, but we only give a time-invariant example here for simplicity.

Fig. 2. An example of trellis graph which satisfies the conditions of Theorem 1.

Since $\mathfrak{r}(A) = \{1, 2, 3\}$ and letting $\chi = \mathfrak{r}(A)$, condition (e) of Theorem 1 is satisfied. One further sets $q = 1$ and calculates that $\bigcup_{\sigma \in \mathcal{S}_k} \sigma = \{1, 2, 3\} = V$. Hence, condition (d) holds too.

The structure of the transition probability in conditions (c)–(e) of Theorem 1 can be described by a trellis graph (Touri & Nedić, 2012). The trellis graph of the random process $\{\mathcal{E}_k\}_{k \geq 1}$ is an infinite directed graph $\mathcal{T} = (\mathcal{V}, \mathcal{E}, \{\sigma_k\}_{k \geq 1})$, where $\mathcal{V}$ is the infinite grid $2^V \times \mathbb{Z}^+$ and

$$\mathcal{E} = \{((\sigma, k), (\sigma', k+1)) \mid \sigma, \sigma' \in 2^V, k \geq 1\}.$$

According to the above definition, the link in a trellis graph is pointed from time k to time $k+1$ if the corresponding transition probability is nonzero.

As shown in Fig. 2, the given trellis graph satisfies the conditions of Theorem 1 for $V = \{1, 2, 3, 4\}$, $\mathfrak{r}(A) = \chi = \{1, 2\}$, and $q = 2$. It should be noted that the weight of each edge in Fig. 2 is dependent on the historic values.

In Theorem 1, we do not require $\{\mathcal{E}_k\}_{k \geq 1}$ to be i.i.d or Markovian. Hence, some popular methods for linear stochastic systems, such as LMI (linear matrix inequality), cannot be easily applied. Even if $\{\mathcal{E}_k\}_{k \geq 1}$ are Markovian, the number of states in the Markovian chain is $|\mathcal{S}| = 2^N$ and hence it generates 2^N LMIs by using the method given in Boukas and Yang (1995). Unfortunately, LMIs with such a huge dimension is very difficult to be solved.

The conditions of Theorem 1 imply the sample space of A_{σ_k} may contain matrices with zero diagonal entries. Traditionally, the positivity of the diagonal entries in the stochastic matrices plays a very important role in random or deterministic consensus, which can be seen in Porfiri and Stilwell (2007), Shi and Johansson (2013), and Tahbaz-Salehi and Jadbabaie (2010). In Theorem 1, we do not require positivity of the diagonal entries in the sample space $\mathfrak{A}$ to guarantee consensus of asynchronous DCA (2).

As simple corollaries of Theorem 1, we consider the following two practical cases of the random process $\{\mathcal{E}_k\}_{k \geq 1}$ which governs the asynchronous iteration:

- (a) A global asynchronous clock: In this case, a global clock determines which node to update, and there is only one node updates at each time. Hence, $\sigma_k \in V$ is the unique node updates at time k .
- (b) Independent asynchronous clocks: In this case, each node has a local clock, such a clock determines the update of the corresponding node independently. Hence, the updated nodes σ_k at time k is a set and it can be decomposed as $\sigma_k = \bigcup_{j=1}^N \theta_k^{(j)}$, where the j th random variable $\theta_k^{(j)} \in \{\{j\}, \emptyset\}$ is associated with the j th local clock. If $\theta_k^{(j)} = \{j\}$, then node j updates at time k . If $\theta_k^{(j)} = \emptyset$, then node j does not update at time k .

The sufficient conditions for almost surely consensus of asynchronous DCA (2) in the above two cases are given as follows.

Theorem 2. The asynchronous DCA (2) generates consensus almost surely if all the following conditions hold:

- (a) $\sigma_k \in V$ and $\mathcal{G}(A)$ is rooted.
- (b) There exists $\alpha > 0$ such that for any $k \geq 1$

$$\mathbb{P}(\sigma_k = j) \geq \alpha, \quad \forall j \in V.$$

Theorem 3. The asynchronous DCA (2) generates consensus almost surely if all the following conditions hold:

- (a) $\mathcal{G}(A)$ is rooted.
- (b) For any $k \geq 1$, $\sigma_k = \bigcup_{j=1}^N \theta_k^{(j)}$, where $\theta_k^{(j)} \in \{\{j\}, \emptyset\}$.
- (c) There exists $\alpha \in (0, 1)$ such that for any $k \geq 1$,
$$\mathbb{P}(\theta_k^{(j)} = \{j\}) \in [\alpha, 1 - \alpha], \quad \forall j \in V.$$

4. Technical skeleton

Due to the limitation of space, we only present the technical skeleton of the proof of Theorem 1 as follows and the details can be found in Chen et al. (2018a).

First, we prove the equivalence between random asynchronous consensus and the convergence of the ergodic coefficient of products of stochastic matrices (see, Lemmas 1 and 2).

Lemma 1. If the asynchronous DCA (2) realizes consensus almost surely under the conditions of Theorem 1, then it realizes consensus uniformly with respect to the region of the initial value $\mathcal{A} = \{x : \Delta(x) \leq 1\}$.

Lemma 2. The asynchronous DCA (2) realizes consensus almost surely if and only if $\lim_{k \rightarrow \infty} \mathbb{P}(\lambda(A_{\sigma(k:1)}) \geq \varepsilon) = 0$ holds for any $\varepsilon > 0$.

Second, we show that the transition probability among $\{\sigma_k\}_{k \geq 1}$ has a special property: the backward probabilities in the trellis graph have a positive infimum (see, Lemma 3).

Lemma 3. If the conditions of Theorem 1 are satisfied, then for any $T > 0$, there exists $\gamma \in (0, 1)$ such that

$$\mathbb{P}(\sigma_{k-1} \mid \sigma_{T:k}) \geq \gamma, \quad \forall T \geq k \geq 2,$$

when $\mathbb{P}(\sigma_{k-1} \mid \sigma_{T:k}) \neq 0$.

Third, based on the result of step 2, the asynchronous consensus problem will be mapped to the problem of random backward walk along a labelled directed cycle (see, Lemmas 4, 5), and the existence of such a cycle is guaranteed by Proposition 1.

Lemma 4. Given a sequence of column stochastic matrices $\{P_k\}_{k=1}^\infty$, if $P_k \geq W$ and $P_k \sim W$ for each $k \geq 1$, and $\mathcal{G}(W^T)$ is rooted with node 1 as the unique root which contains a self-loop, then there exist $c_0 > 0$ and $\beta \in (0, 1)$ such that $\|P_k \cdots P_2 P_1 - \mathbf{e}_1 \mathbf{1}^T\| \leq c_0 \beta^k$, where $\mathbf{e}_1$ is the elementary unit vector with the 1st entry being 1 and the other entries being 0.

Lemma 5 (Random Walk Along a Labelled Directed Cycle). Given a labelled directed cycle $\mathcal{C}$ with the set of nodes $V = \{1, 2, \dots, l\}$ and the corresponding set of labels $V_l = \{\text{label}(i) : i \in V\}$, we define the following random walk $\mathcal{W}$ along this cycle: Given two nodes moving along the cycle $\mathcal{C}$, the positions of them at time k are denoted by i_k and j_k with the corresponding historic values $W_k = (i_\tau, j_\tau)_{\tau=1}^k$, the transition from (i_k, j_k) to (i_{k+1}, j_{k+1}) is described by:

When $\text{label}(i_k) \neq \text{label}(j_k)$, it holds that

$$\begin{aligned} \mathbb{P}((i_{k+1}, j_{k+1}) = (i_k, j') \mid W_k) &\geq \gamma, \\ \mathbb{P}((i_{k+1}, j_{k+1}) = (i', j_k) \mid W_k) &\geq \gamma, \\ \mathbb{P}((i_{k+1}, j_{k+1}) \in \{(i_k, j_k), (i', j')\} \mid W_k) &\geq \gamma, \end{aligned}$$

where $i' \rightarrow i_k$ and $j' \rightarrow j_k$ in $\mathcal{C}$, and

$$\sum_{i \in \{i_k, i'\}, j \in \{j_k, j'\}} \mathbb{P}((i_{k+1}, j_{k+1}) = (i, j) \mid W_k) = 1;$$

When $\text{label}(i_k) = \text{label}(j_k)$, it holds

$$\mathbb{P}((i_{k+1}, j_{k+1}) = (i_k, j_k) \mid W_k) = 1.$$

For the above random walk $\mathcal{W}$, there exists an integer $k^* > 0$ and real numbers $\mu_k \in (0, 1)$ such that

$$\mathbb{P}(\text{label}(i_{k+1}) = \text{label}(j_{k+1}) \mid W_k) \geq \mu_k$$

holds for any $k \geq k^*$.

Fourth, we show that the random product of asynchronous matrix sequence has a nonzero probability to be scrambling (see, Lemmas 6, 7).

Lemma 6. If the conditions of Theorem 1 are satisfied, then given any two nodes $i, j \in V$ ($i \neq j$), there exists T^* such that $\mathbb{P}(\mathcal{N}(A_{\sigma_{T:1}}, i) \cap \mathcal{N}(A_{\sigma_{T:1}}, j) \neq \emptyset) \geq h_T \in (0, 1)$ holds for any $T \geq T^*$.

Lemma 7. If the conditions of Theorem 1 are satisfied, then $\mathbb{P}(\lambda(A_{\sigma_{M:1}}) \leq 1 - \delta^M) \geq h_T^{\frac{N(N-1)}{2}}$, where $M = \frac{(N-1)N}{2}T$, $T \geq T^*$, δ is the minimal positive entry of A , T^* and h_T are given in Lemma 6.

Finally, the proof of Theorem 1 can be obtained by partitioning the matrix product $A_{M:1}$ to many subproducts with each having a nonzero probability to be scrambling. By using the basic properties of ergodic coefficient, the convergence can be derived if scrambling matrices appear infinitely many times.

The proof of the above lemmas and the proof of Theorem 1 are given in Chen et al. (2018a) due to page limit. In particular, both Theorems 2 and 3 can be directly obtained from Theorem 1, hence the proof of them has been omitted. Moreover, one can refer to Chen et al. (2018a) for more corollaries of Theorem 1.

5. Numerical examples

In this section, we give two numerical examples to verify the effectiveness of Theorems 2 and 3.

Consider the following stochastic matrix

$$A = \begin{pmatrix} \mathbb{O}_{3 \times 3} & A_1 \\ A_2 & \mathbb{O}_{3 \times 3} \end{pmatrix} \quad (6)$$

$$A_1 = \begin{pmatrix} 0.5 & 0 & 0.5 \\ 0 & 0.5 & 0.5 \\ 1 & 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $\mathcal{G}(A)$ is strongly connected. According to Seneta (1981), one can see A is not an SIA matrix. As shown in Fig. 3, the synchronous DCA (1) cannot realize consensus with the above A . However, when we implement the asynchronous DCA (2) with the same matrix A , the dynamics becomes quite different:

Firstly, suppose DCA (2) has a global asynchronous clock. For any $1 \leq j \leq 6$, one sets $\mathbb{P}(\sigma_k = j) = 1/6$ and the conditions of Theorem 2 are satisfied. As shown in Fig. 4, the agents realize consensus almost surely when the initial values are chosen randomly from $[-1, 1]$.

Next, suppose DCA (2) has independent local clocks. For any $1 \leq j \leq 6$, one sets $\mathbb{P}(\theta_k^{(j)} = j) = \mathbb{P}(\theta_k^{(j)} = \phi) = \frac{1}{2}$ and the conditions of Theorem 3 are satisfied. As shown in Fig. 5, the agents realize consensus almost surely when the initial values are chosen randomly from $[-1, 1]$.

6. Conclusions

This paper investigated the asynchronous iteration problem of distributed coordination algorithms in the stochastic setting. By asynchronous iteration we mean only a fraction of nodes update their states at a given time instant. We have found that

Fig. 3. Synchronous DCA (1) with A given in (6): cannot realize consensus.

Fig. 4. Asynchronous DCA (2) with A given in (6): a global asynchronous clock.

Fig. 5. Asynchronous DCA (2) with A given in (6): independent asynchronous clocks.

the topology only needs to be rooted for consensus of DCA with asynchronous iteration in the given stochastic setting, which is in sharp contrast to the deterministic setting where the topology should be rooted and aperiodic. In the future, we will apply the proposed DCA with random asynchronous iteration to resolve practical engineering problems, such as clock synchronization in wireless sensor networks.

Acknowledgements

This work was supported in part by the National Key Research and Development Program of China under Grant 2016YFB0800401, in part by the National Natural Science Foundation of China under Grants 61773319, 61603071, 61621003, 61532020, in part by the Fundamental Research Funds for the Central Universities under Grants DUT19ZD103, JBK190502, in part by the Youth Star of Dalian Science and Technology, China (2018RQ51),

in part by the European Research Council (ERC-CoG-771687), and in part by the Netherlands Organization for Scientific Research (NWO-vidi-14134).

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